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PATH - INDEPENDENT INTEGRALS IN FINITE ELASTICITY AND INELASTICITY, WITH BODY FORCES, INERTIA, AND ARBITRARY CRACK - FACE CONDITIONS.

by

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July 1981

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PATH - INDEPENDENT INTEGRALS IN FINITE ELASTICITY AND INELASTICITY, WITH BODY FORCES, INERTIA, AND ARBITRARY CRACK - FACE CONDITIONS:

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Abstract:

In this paper, certain path-independent integrals, of relevance in the presence of cracks, in elastic and inelastic solids are considered. The hypothesized material constitutive properties include: (i) finite and infinitesimal elasticity, (ii) rate-independent incremental flow theory of elasto-plasticity, and (iii) rate-sensitive behaviour including elasto-viscoplasticity, and creep. In each case, finite deformations are considered, along with the effects of body forces, material acceleration, and arbitrary traction/displacement conditions on the crack-face. Also the physical interpretations of each of the integrals either in terms of crack-tip energy release rates or simply energy-rate differences in two comparison cracked-bodies are explored. Several differences between the results in the present work and those currently considered well established in literature are pointed out and discussed.

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Introduction:

Since the pioneering work of Eshelby [1], and independent discovery of Rice [2], innumerable number of papers have appeared in literature concerning path-independent integrals (for the most part however, concerning the so-called J-integral) and their application to mechanics of fracture. No attempt is made herein at a comprehensive survey of this still burgeoning literature. Of particular relevance to the present work are the important studies, on conservation laws in finite and infinitesimal elasticity, by Knowles and Sternberg [3], and the interpretation of these in the context of the mechanics of cracks and notches in 2-dimensional bodies by Budiansky and Rice [4]. It is noted that the studies in [1-4] are restricted to elasto-statics, and the crack-extension considered, if any, is of quasi-static nature i.e., inertia is considered negligible. As is often noted in literature, the so-called J-integral in elastostatic crack mechanics is in fact one of the components of a vector-integral, and its relevance is in the context of incipient, self-similar, crack-extension.

Eventhough the so-called J-integral [1,2] was intended to be applicable to finite or infinitesimal elasticity, its use had been extended, in several works in the past twelve years, far beyond the range of its apparent theoretical validity. Invoking "deformation theory of plasticity" and/or "proportional loading", it was used in the context of initiation of "Mode I" growth of a crack in an elasto-plastic body. A wide class of literature has also grown around the concept of the so-called "J_{IC} test". Rice [5], in an article which appeared in 1976, and which is perhaps accurately representative even today, succinctly summarized a wide body of literature pertaining to the use of J in the context of elasto-plasticity. Later, based more or less on empirical reasoning, J and the rate of change of J with crack length ("dJ/da") were postulated to be "valid" parameters characterizing even stable crack

growth in ductile materials [6], wherein crack-growth necessarily implies unloading. However, the fundamental concept of a pat independent integral and its physical meaning, if any, in the context of an (even rate-independent) incremental flow theory of plasticity yet remains to be explored.

In the class of problems generally characterized as belonging to the domain of "dynamic fracture mechanics", integral relations quantifying the rate of energy-release to a propagating crack-tip in a plane linear elastic body undergoing infinitesimal deformation were presented by Freund [7,8], who also succinctly summarized the pertinent work of Atkinson, Eshelby, Achenbach, Sih, and others. The path-independency, if any, of such integrals for energy-release rates even in linear elastodynamic crack propagation yet remain to be understood.

To the author's knowledge, no work has been reported concerning path-independent integrals, which may characterize the severity of the conditions near the crack-tip, in materials characterized by rate-sensitive inelastic constitutive laws, such as, for instance, viscoplasticity and creep. However, in the case of pure steady-state creep characterized by a power law (of the type $\dot{\varepsilon}\sim\sigma^n$), an integral $\dot{J}$ (or c*) which is entirely similar to the J[1,2] for pure power-law hardening materials ($\varepsilon\sim\sigma^n$) was introduced by Goldman and Hutchinson [9] and Landes and Begley [10], based on the observed similarity of the constitutive laws in the respective cases (i.e., $\dot{\varepsilon}$ instead of ε, etc.). However, the physical interpretation, if any, of c* appears not to have been fully explored.

The present work represents a modest effort at a re-examination of path-independent integrals, and their relevance to mechanics of cracks, in elastic as well as inelastic solids. The postulated material behaviour includes the cases of finite elasticity, rate-independent incremental flow theory of plasticity, and rate-sensitive behaviour such as visco-plasticity and creep. In

each case finite deformations are considered, along with the effects of body forces, material acceleration, and arbitrary traction/displacement conditions on the crack-face.

We start by considering the case of finite elasticity and attempt to generalize the conservation laws given by Knowles and Sternberg [3] to the case when body forces, inertia, and arbitrary crack-face conditions are accounted for. From these conservation laws, we derive a path-independent vector integral of relevance to fracture-mechanics. In this process, we re-examine the fundamental notion of "path-independence", and the attendant mathematical and physical reasoning, as originally propounded by Rice [2], and the results presented later by Budiansky and Rice [4]. We specialize the obtained results to the case of linear elastodynamic crack-propagation. To understand the physical meaning of the "path-independent" integral vector, we make an independent study of the expression for the rate of energy-release in elastodynamic crack-propagation. Several differences between the results in the present work and those currently considered well-established in literature are noted and discussed.

In the second part of the paper, we consider conservation laws, and the attendant path-independent integrals, in the incremental theory of finite-deformation, rate-independent, classical elastoplasticity. Once again we include body forces, inertia, and general crack-face conditions in the discussion. Also, we explore the physical meaning of the path-independent integral incremental-vector, in the case of elasto-plasticity.

In the final part of the paper we consider finite strain rate-sensitive inelasticity characterized by a elasto-viscoplastic constitutive law of the type proposed by Perzyna [11]. We also treat the well-known Norton's power-law type of steady state creep as a special case. We point out certain "incremental" integrals, which are: path-independent in a limited sense in the case

elasto-viscoplastic strains, and strictly path-independent in the presence of combined elastic and creep strain rates. We explore the physical meaning of these integrals as well.

<u>Notation</u>:

For convenience to the reader, we summarize the notations employed in the present as follows:

$(\underline{\ })$		under symbol denotes a vector
$(\sim)$		under symbol denotes a second-order tensor
$\underline{a} = \underset{\sim}{A}.\underline{b}$		implies $a^i = A^{i\cdot}_{\cdot j} b^j$
$\underset{\sim}{C} = \underset{\sim}{A}.\underset{\sim}{B}$		implies $C^{ij} = A^{i\cdot}_{\cdot K} B^{Kj}$
$\underset{\sim}{A}:\underset{\sim}{B}$		implies the trace: $A^{ij}B_{ij}$
$\underset{\sim}{A} = A^{ij}\underline{g}_i\underline{g}_j$		a second-order tensor in dyadic notation
$\underline{\nabla}_p = \underline{G}^L \dfrac{\partial}{\partial \xi^L}$		gradient operator
$\underline{u}, \Delta\underline{u}$		displacement vector and its increment
$\underset{\sim}{t}$	:	first Piola-Kirchhoff stress
$\underset{\sim}{s}$	:	second Piola-Kirchhoff stress
$\underset{\sim}{\tau}$	:	Cauchy stress
$\underset{\sim}{\sigma}$	:	Kirchhoff stress
$\underset{\sim}{\hat{\dot{\sigma}}}$	:	co-rotational rate of Kirchhoff stress

Finite Elasticity:

Consider a solid body with an initial undeformed configuration B and a deformed configuration b. Let the coordinates of a point P in B be ξ^J, with primary base vectors $\underline{G}_J$. The material particle at P in B is assumed to have moved to the point p in 'b'. Let 'b' be defined by another set of arbitrary curvilinear spatial coordinates η^i, or by convected coordinates ξ^J, with base vectors $\underline{g}_i$ and $\underline{g}_J$ respectively. Let the vector of displacement of the particle from P to p be $\underline{u} = u^J \underline{G}_J$. The deformation gradient tensor $\underset{\sim}{F}$ is:

$$\underset{\sim}{F} = F^{i\cdot}_{\cdot J} \underline{g}_i \underline{G}^J = \underline{g}_J \underline{G}^J$$

$$= \frac{\partial \eta^i}{\partial \xi^J} \underline{g}_i \underline{G}^J \tag{1}$$

wherein, use has been made of the dyadic notation. In terms of $\underline{u}$, we can write:

$$\underset{\sim}{F} = F^{L\cdot}_{\cdot K} \underline{G}_L \underline{G}^K = (\delta^L_K + u^L_{,K}) \underline{G}_L \underline{G}^K \equiv \underset{\sim}{I}(P) + \underset{\sim}{e} \tag{2}$$

where $\underset{\sim}{I}(P)$ is the identity (metric) tensor at P, and $(\ )_{,K}$ denotes a covariant derivative w.r.t. ξ^K at P. We see that the displacement gradient tensor $\underset{\sim}{e}$ is:

$$\underset{\sim}{e} = u^L_{,K} \underline{G}_L \underline{G}^K. \tag{3}$$

Let $\underset{\sim}{\tau}$ be the tensor of true or Cauchy stress at p, which measures tractions on an oriented surface $(da\underline{n})$ at p. If the image of $(da\underline{n})$ in the undeformed configuration is $(dA\underline{N})$ at P, we define the first and second Piola-Kirchhoff stress tensors, denoted here by $\underset{\sim}{t}$ and $\underset{\sim}{s}$ respectively, as:

$$(dA\underline{N}) \cdot \underset{\sim}{t} = da\underline{n} \cdot \underset{\sim}{\tau} \tag{4}$$

$$(dA\underline{N}) \cdot \underset{\sim}{s} = (da\underline{n}) \cdot (\underset{\sim}{\tau} \cdot \underset{\sim}{F}^{-T}) \tag{5}$$

where $(\underset{\sim}{\ })^{-1}$ denotes an inverse of a tensor, and $(\ )^T$ denotes a transpose. Using the geometrical relation

$$(da\underline{n}) = J(da\underline{N}) \cdot \underset{\sim}{F}^{-1} \tag{6}$$

where J is the absolute determinant of $\underset{\sim}{F}$ (or the determinant of the matrix $[F^{L\cdot}_{\cdot K}]$). Eqs. (4) and (5) lead to:

$$\underset{\sim}{t} = J\underset{\sim}{F}^{-1} \cdot \underset{\sim}{\tau} \equiv \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \tag{7}$$

$$\underset{\sim}{s} = J\underset{\sim}{F}^{-1} \cdot \underset{\sim}{\tau} \cdot \underset{\sim}{F}^{-T} = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{-T} = \underset{\sim}{t} \cdot \underset{\sim}{F}^{-T} \tag{8}$$

where, $\underset{\sim}{\sigma} = J\underset{\sim}{\tau}$ is the Kirchhoff stress.

Noting that:

$$\underset{\sim}{F}^{-1} = (F^{-1})^{J\cdot}_{\cdot i}\underline{G}_J\underline{g}^i = \frac{\partial\xi^J}{\partial\eta^i}\underline{G}_J\underline{g}^i = \underline{G}_J\underline{g}^J \tag{9}$$

and

$$\underset{\sim}{\tau} = \tau^{i\cdot}_{\cdot j}g_i g^j = \tau^{K*\cdot}_{\cdot L*}g_K g^L \tag{10}$$

we have:

$$\underset{\sim}{t} = J\frac{\partial\xi^J}{\partial\eta^i}\tau^{i\cdot}_{\cdot n}\underline{G}_J\underline{g}^n = J\tau^{K*\cdot}_{\cdot L*}\underline{G}_K\underline{g}^L \tag{11a}$$

and

$$\underset{\sim}{s} = J\frac{\partial\xi^J}{\partial\eta^m}\tau^{mn}\frac{\partial\xi^L}{\partial\eta^n}\underline{G}_J\underline{G}_L. \tag{11b}$$

Let W be the strain-energy density measured per unit volume at P in B. Then it is known that:

$$\frac{\partial W}{\partial \underset{\sim}{F}} = \frac{\partial W}{\partial \underset{\sim}{e}} = \underset{\sim}{t}^T \tag{12}$$

and

$$\frac{\partial W}{\partial \underset{\sim}{\gamma}} = 2\frac{\partial W}{\partial \underset{\sim}{C}} = \underset{\sim}{s} \tag{13}$$

where $\underset{\sim}{C} = \underset{\sim}{F}^T.\underset{\sim}{F}$ and $\underset{\sim}{\gamma} = \frac{1}{2}[\underset{\sim}{C}-\underset{\sim}{I}(P)]$ is the Green-Lagrange strain tensor. The equations of linear (LMB) and angular (AMB) momentum balance conditions are:

(LMB) : $\underline{\nabla}_P \cdot \underset{\sim}{t} + \rho_o(\underline{f}-\underline{a}) = 0$

$$\text{or} \quad \underline{\nabla}_P \cdot [\underset{\sim}{s}.\underset{\sim}{F}^T] + \rho_o(\underline{f}-\underline{a}) = 0 \tag{14a,b}$$

$$(\text{AMB}) \quad : \quad \underset{\sim}{F} \cdot \underset{\sim}{t} = \underset{\sim}{t}^T \cdot \underset{\sim}{F}^T$$

$$\text{or} \quad \underset{\sim}{s} = \underset{\sim}{s}^T. \tag{15a,b}$$

Compatibility:

$$\underset{\sim}{F} = \underset{\sim}{I}(P) + \underset{\sim}{e} = (\delta^L_K + u^L_{,K}) \, \underline{G}_L \underline{G}^K$$

$$\text{or } \underset{\sim}{\gamma} = \frac{1}{2}[u_{L,K} + u_{K,L} + u^M_{,L} u_{M,K}] \, \underline{G}^L \underline{G}^K. \tag{16a,b}$$

Boundary Conditions:

$$\underline{N} \cdot \underset{\sim}{t} = \underline{N} \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^T) = \bar{\underline{t}} \text{ at } S_t \tag{17a}$$

$$\underline{u} = \bar{\underline{u}} \text{ at } S_u. \tag{17b}$$

For our present purposes, we restate Eq. (17b) in a bit more ambiguous, but still physically meaningful form as:

$$\underset{\sim}{F} = \bar{\underset{\sim}{F}} \text{ at } S_F. \tag{17c}$$

The AMB condition, Eq. (15a) is automatically satisfied provided W is a frame-indifferent function of $\underset{\sim}{F}$ [i.e., for instance, when W is a function of $\underset{\sim}{F}$ only through $W(\underset{\sim}{F}) = W(\underset{\sim}{C}) = W(\underset{\sim}{F}^T \cdot \underset{\sim}{F})$] and $\underset{\sim}{t}$ is defined through Eq. (12). In Eq. (14), ρ_o is mass/unit volume at P in B, $\underline{f}$ are arbitrary body forces per unit mass, and $\underline{a}$ is the absolute acceleration of the material particle [$\underline{a} = a_L \underline{G}^L = (d^2 u_L/dt^2) \, \underline{G}^L \equiv \ddot{u}_L \underline{G}^L$]. Likewise, in Eq. (17a), $\bar{\underline{t}}$ are prescribed tractions per unit undeformed area S_t of B. If $\underset{\sim}{t}$, $\underset{\sim}{s}$, $\underset{\sim}{F}$, $\underset{\sim}{e}$, $\underline{u}$ are fields that satisfy Eqs. (12 through 17), we now show that for any close volume V, which is free from any singularities or other defects, in the configuration B, the following identities* are valid:

$$\underline{0} = \int_V [\underline{\nabla}_P W - \underline{\nabla}_P \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F}) - \rho_o (\underline{f} - \underline{a}) \cdot \underset{\sim}{F}] dv + \int_{S_t} (\underline{N} \cdot \underset{\sim}{t} - \bar{\underline{t}}) \cdot \underset{\sim}{F} ds + \int_{S_F} \underline{N} \cdot \underset{\sim}{t} \cdot (\underset{\sim}{F} - \bar{\underset{\sim}{F}}) ds \tag{18}$$

$$\underline{0} = \int_V [\underline{\nabla}_P W - \underline{\nabla}_P \cdot (\underset{\sim}{s} \cdot \underset{\sim}{C}) - \rho_o (\underline{f} - \underline{a}) \cdot \underset{\sim}{F}] dV + \int_{S_t} (\underline{N} \cdot \underset{\sim}{s} \cdot \underset{\sim}{F}^T - \bar{\underline{t}}) \cdot \underset{\sim}{F} ds + \int_{S_F} \underline{N} \cdot \underset{\sim}{s} \cdot \underset{\sim}{F}^T \cdot (\underset{\sim}{F} - \bar{\underset{\sim}{F}}) dS \tag{19}$$

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* These represent a generalization of the conservation law given in [3], to the case when body forces, inertia, and arbitrary crack-face conditions are considered.

In the above, we made the assumption that the material is homogeneous, i.e., W is a function of the location P in B only by virtue of the fact that F is, in general, a function of P. The proof of Eq. (18) is evident from the following:

$$\underline{\nabla}_P W = \underline{G}^L \left(\frac{\partial W}{\partial \xi^L}\right) = \underline{G}^L \left[\frac{\partial W}{\partial \underset{\sim}{F}} : \frac{\partial \underset{\sim}{F}}{\partial \xi^L}\right]$$

$$= \underline{G}^L \left[\underset{\sim}{t}^T : (\underset{\sim}{F})_{,L}\right] = \underline{G}^L \left[t^{MN} \underline{G}_N \underline{G}_M : (F_{KJ})_{,L} \underline{G}^K \underline{G}^J\right]$$

$$= \underline{G}^L \left[t^{MN} (F_{KJ})_{,L} (\underline{G}_N \cdot \underline{G}^K)(\underline{G}_M \cdot \underline{G}^J)\right] \equiv \underline{G}^L \left[t^{MN} (F_{NM})_{,L}\right] \tag{20}$$

where $(\ )_{,L}$ denotes covariant differentiation w.r.t. ξ^L. And,

$$\underline{\nabla}_P \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F}) = \underline{G}^L \cdot \frac{\partial}{\partial \xi^L}\left[t^{MN} F_{NK} \underline{G}_M \underline{G}^K\right]$$

$$= (t^{MN} F_{NK})_{,M} \underline{G}^K \equiv (t^{MN} F_{NL})_{,M} \underline{G}^L$$

$$= (t^{MN}{}_{,M} F_{NL} + t^{MN} F_{NL,M}) \underline{G}^L. \tag{21}$$

But $$F_{NL,M} = (G_{NL} + u_{N,L})_{,M} \equiv (u_{N,L})_{,M} = (u_{N,M})_{,L}$$

$$\equiv (G_{NM} + u_{N,M})_{,L} = (F_{NM})_{,L} \tag{22}$$

since, the metric tensor behaves as a constant during covariant differentiation. Thus, equation (21) can be written as:

$$\underline{\nabla}_P \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F}) = (t^{MN}{}_{,M} F_{NL} + t^{MN} F_{NM,L}) \underline{G}^L$$

$$= (\underline{\nabla}_P \cdot \underset{\sim}{t}) \cdot \underset{\sim}{F} + [\underset{\sim}{t}^T : (\underset{\sim}{F})_{,L}] \underline{G}^L. \tag{23}$$

Upon using Eqs. (20), (23), (14a), (17a), and (17c), the validity of Eq. (18) is immediately evident. When it is noted that $\underset{\sim}{t} = \underset{\sim}{s} \cdot \underset{\sim}{F}^T$, and $\underset{\sim}{F}^T \cdot \underset{\sim}{F} = \underset{\sim}{C}$, the proof of Eq. (19) is apparent. An independent proof, follows from:

$$\underline{\nabla}_P W = \underline{G}^L [\underset{\sim}{s} : (\underset{\sim}{\gamma})_{,L}] = \underline{G}^L \{ \underset{\sim}{s} : \tfrac{1}{2} [(\underset{\sim}{F}^T)_{,L} \cdot \underset{\sim}{F} + \underset{\sim}{F}^T \cdot (\underset{\sim}{F})_{,L}] \}$$

$$= \underline{G}^L\{\underset{\sim}{s} : [(\underset{\sim}{F})^T,_L \cdot \underset{\sim}{F})]\} = \underline{G}^L\{(\underset{\sim}{s} \cdot \underset{\sim}{F}^T) : (\underset{\sim}{F}^T),_L\}. \tag{24}$$

Since $\underset{\sim}{s}$ is a symmetric tensor. Further

$$\underline{\nabla}_p \cdot [\underset{\sim}{s} \cdot \underset{\sim}{C}] = \underline{\nabla}_p \cdot [\underset{\sim}{s} \cdot \underset{\sim}{F}^T \cdot \underset{\sim}{F}] \equiv [\underline{\nabla}_p \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^T)] \cdot \underset{\sim}{F} + (\underset{\sim}{s} \cdot \underset{\sim}{F}^T : (\underset{\sim}{F}),_L \underline{G}^L$$

$$= [\underline{\nabla}_p \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^T)] \cdot \underset{\sim}{F} + (\underset{\sim}{s} \cdot \underset{\sim}{F}^T) : (\underset{\sim}{F}^T),_L \underline{G}^L. \tag{25}$$

From (24), (25), (14b), (17a) and (17c), the validity of Eq. (19) can be noted.

Using the Green-Gauss theorem, we can write Eqs. (18) and (19) in the form:

$$\underline{0} = \int_{\partial V} [\underline{N}W - \underline{N} \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F})]\, ds - \int_V \rho_o(\underline{f}-\underline{a}) \cdot \underset{\sim}{F}dv + \int_{S_t} (\underline{N} \cdot \underset{\sim}{t} - \underline{\bar{t}}) \cdot \underset{\sim}{F}ds$$

$$+ \int_{S_F} \underline{N} \cdot \underset{\sim}{t} \cdot (\underset{\sim}{F} - \bar{\underset{\sim}{F}})\, ds \tag{26}$$

$$= \int_{\partial V} \underline{N}W - \underline{N} \cdot (\underset{\sim}{s} \cdot \underset{\sim}{C}) - \int_V \rho_o(\underline{f}-\underline{a}) \cdot \underset{\sim}{F}dV + \int_{S_t} (\underline{N} \cdot \underset{\sim}{t} - \underline{\bar{t}}) \cdot \underset{\sim}{F}ds$$

$$+ \int_{S_F} \underline{N} \cdot \underset{\sim}{t} \cdot (\underset{\sim}{F} - \bar{\underset{\sim}{F}})\, ds \tag{27}$$

Where ∂V is the surface of V, and we assume that

$$\partial V = S_t + S_F + S_i. \tag{28}$$

Thus, Eq. (26) can be written as:

$$\underline{0} = \int_{S_i} [\underline{N}W - \underline{N} \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F})]\, ds - \int_V \rho_o(\underline{f}-\underline{a}) \cdot \underset{\sim}{F}dv$$

$$+ \int_{S_t} [\underline{N}W - \underline{\bar{t}} \cdot \underset{\sim}{F}]\, ds + \int_{S_F} (\underline{N}W - \underline{N} \cdot \underset{\sim}{t} \cdot \bar{\underset{\sim}{F}})\, ds. \tag{29}$$

Eventhough Eq. (29) is applicable to a general 3-dimensional case, an illustration of the 2-dimensional case is given in Fig. 1, wherein S_i, S_F, and S_t are depicted, and $\underline{N}$ is an unit 'outward' normal to ∂V as shown also in Fig. 1.

Suppose we consider the cartesian coordinate system: x_1 along the crack surface, x_2 normal to the crack face, and x_3 along the crack front, and consider the component along the x_1 direction of the vector identity, Eq. (29). Let the problem be also a <u>special case</u> in which (i) the crack-faces are free of tractions and any imposed displacement conditions, (ii) the body forces $\underline{f}$ are zero, and (iii) the inertia is negligible, i.e., the problem is one of elastostatics. In this case, since $N_1 = 0$ along the face, we have from Eq. (29),

$$\int_{\Gamma_{BCD}} (N_1 W - N_j t_{jk} F_{kl})\, ds = \int_{\Gamma_{AFE}} (N_1 W - N_j t_{jk} F_{kl})\, ds. \qquad (30)$$

It is this sense of path-independence of the integral on Γ_{BCD}, and the associated physical interpretation of the integral, that were essentially presented by [2]. However, in the general case, i.e., the case in which: any of the conditions (i) - (iii) above are not satisfied, and, in addition, the components in x_2 and x_3 directions of the vector identity (29) are also of interest, we need to re-examine the above path-independence. For purposes of understanding the above general case, let us apply*the result in Eq. (29) to a volume $V-V_\varepsilon$ as shown in Fig. 2a, for a 2-dimensional problem and for 3-dimensional problem in Fig. 2b. For purposes of clarity in presentation, let us consider, without loss of generality, the 2-dimensional case. In view of Eq. (29) we then have, referring to Fig. 2a,

$$\int_{\Gamma_{561}} [\underline{N}W-\underline{N}.(\underline{t}.\underline{F})]\, ds + \int_{\Gamma_{234}} [\underline{N}W-\underline{N}.(\underline{t}.\underline{F})]\, ds - \int_{V-V_\varepsilon} \rho_o(\underline{f}-\underline{a}) . \underline{F}dv$$

$$+ \int_{\Gamma'_{45}} [\underline{N}W-\bar{\underline{t}}.\underline{F}]\, ds + \int_{\Gamma''_{45}} [\underline{N}W-\underline{N}.\underline{t}.\bar{\underline{F}}]\, ds$$

$$+ \int_{\Gamma'_{12}} [\underline{N}W-\bar{\underline{t}}.\underline{F}]\, ds + \int_{\Gamma''_{12}} [\underline{N}W-\underline{N}.\underline{t}.\bar{\underline{F}}]\, ds = 0 \qquad (31)$$

* Note that the divergence theorem cannot, in general, be applied when the volume integral contains a non-integrable singularity. Thus, when volume V includes the crack-tip, referring to Eq. (18), $\nabla_p W$ may be of order r^{-2} near the crack-tip (front) and hence non-integrable. Thus the divergence theorem cannot be applied to (Eq. 18) in the case of V, but only in the case of $V-V_\varepsilon$ in the limit $\varepsilon \rightarrow 0$.

where Γ_{561} is the contour with a unit normal directed inwards (into) $V-V_\varepsilon$ as in Fig. 2a. Γ'_{45} and Γ''_{45} are portions of Γ_{45} where tractions and displacements, respectively, are applied. Similar definitions apply to Γ''_{12} and Γ''_{12}. Since the crack is mathematically a surface of discontinuity, it is seen that $\underline{N}^-$, which is a unit outward normal to Γ_{45} is equal to the negative of $\underline{N}^+$, which is the outward normal to Γ_{12}. We now write:

$$\int_{V-V_\varepsilon} \rho_o \underline{a}.\underset{\sim}{F}dv = \int_{V-V_\varepsilon} \left\{\frac{d}{dt}[\rho_o\underline{\dot{u}}.\underset{\sim}{F}] - \underline{\nabla}_P (\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}})\right\} dv$$

$$= \int_{V-V_\varepsilon} \left\{\frac{d}{dt}[\rho_o\underline{\dot{u}}.\underset{\sim}{F}] - \underline{\nabla}_P \cdot [(\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}})\underset{\sim}{I}]\right\} dv. \tag{32}$$

Using the divergence theorem, we further write:

$$-\int_{V-V_\varepsilon} \underline{\nabla}_P (\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}}) dv = -\int_{\Gamma_{561}} \underline{N} (\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}}) ds - \int_{\Gamma_{234}} \underline{N} (\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}})$$

$$- \int_{\Gamma_{12}} \underline{N}^+[(\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}})^+ - (\frac{1}{2}\rho_o\underline{\dot{u}}.\underline{\dot{u}})^-]ds. \tag{33}$$

Using Eqs. (32) and (33) in Eq. (31), we write:

$$\int_{\Gamma_{234}} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t}.\underset{\sim}{F})] ds - \int_{V-V_\varepsilon} \rho_o\underline{f} \cdot \underset{\sim}{F}dv + \int_{V-V_\varepsilon} \left\{\frac{d}{dt}(\rho_o\underline{\dot{u}}.\underset{\sim}{F})\right\} dv$$

$$+ \int_{\Gamma_{12}} \underline{N}^+ [W^+-W^-) - (T^+-T^-)] ds - \int_{\Gamma'_{12}} [(\underline{\bar{t}}.\underset{\sim}{F})^+ + (\underline{\bar{t}}.\underset{\sim}{F})^-] ds$$

$$- \int_{\Gamma''_{12}} \underline{N}^+ \cdot [(\underset{\sim}{t}.\underset{\sim}{\bar{F}})^+ - (\underset{\sim}{t}.\underset{\sim}{\bar{F}})^-] ds$$

$$= \int_{\Gamma_\varepsilon} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t}.\underset{\sim}{F})] ds. \tag{34}$$

Note that in the above, considering, without loss of generality, the segment 12 to be of the same length as the segment 45, the integrals on Γ_{45} and Γ_{12} in Eq. (31) have been combined into a single integral on Γ_{12} alone, with

appropriate sign changes in the unit normal being accounted for. Further, the notation for the kinetic energy density, $T \equiv (1/2)\rho_o \underline{\dot{u}}.\underline{\dot{u}}$ has been employed: and Γ_ε is now equal to Γ_{165} with a unit normal acting inwards (into) $(V-V_\varepsilon)$ as shown in Fig. 2a.

We now consider Eq. (34) in the limit when $\varepsilon \to 0$, i.e. the volume V_ε, and the contour Γ_ε shrink to zero. First consider the right hand side of Eq. (34). Let Γ_ε, in the 2-dimensional problem, be a circle of radius ε. (In 3-dimensional problems, we let Γ_ε be a circular cylindrical surface of radius ε, with the axis of the cylinder being the crack front). Then,

$$\underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t}.\underset{\sim}{F})]\, ds$$

$$= \underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{-\Pi}^{+\Pi} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t}.\underset{\sim}{F})]_{(r=\varepsilon)} \varepsilon d\theta . \tag{35}$$

It is seen that the right-hand side of Eq. (35), for a sharp crack, would vanish identically unless W (and by dimensional considerations, $\underset{\sim}{t}.\underset{\sim}{F}$), and T all have singularities of the type $r^{-\alpha}$ (with r,θ being polar-coordinates centered at the crack-tip) where $\alpha > 1$. However, since the <u>total</u> strain and kinetic energies in a small core region near the crack-tip must be finite, it is seen that α must be equal to 1. Since W is a nonlinear function of $\underset{\sim}{F}$, $\underset{\sim}{F}$ can be expected to have a singularity of type $r^{-\beta}$ with $\beta<1$; likewise $\underset{\sim}{t}$ may have a singularity of type $r^{-\delta}$, such that $\beta+\delta=1$. It is worth noting that available solutions in linear elastodynamic crack propagation indicate that the <u>material</u> time derivative $\underline{\dot{u}}$ (or <u>absolute</u> velocity of a material particle) may have a singularity of $r^{-1/2}$. On the other hand, if the crack is stationary, even when the dynamic effects are accounted for, $\underline{\dot{u}}$ varies as $r^{+1/2}$ [8]. Thus, in general, in a dynamic crack propagation problem, the term on R.H.S. of Eq. (34) is <u>non-zero</u> for a sharp

crack, even in the limit as $\varepsilon \to 0$. <u>We denote this non-zero limit as the vector</u> $\underline{J}$. Thus, from Eqs. (34) and (35) we have:

$$\int_{\Gamma_{234}} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F})]\, ds + \underset{\varepsilon \to 0}{\mathrm{Lt}} \Big\{ - \int_{V-V_\varepsilon} \rho_o \underline{f} \cdot \underset{\sim}{F} dv + \int_{V-V_\varepsilon} \frac{d}{dt} (\rho_o \underline{\dot{u}} \cdot \underset{\sim}{F})\, dv$$

$$+ \int_{\Gamma_{12}} \underline{N}^+ [(W^+ - W^-) - (T^+ - T^-)]\, ds - \int_{\Gamma'_{12}} [(\underline{\bar{t}} \cdot \underset{\sim}{F})^+ + (\underline{\bar{t}} \cdot \underset{\sim}{F})^-]\, ds$$

$$- \int_{\Gamma''_{12}} \underline{N}^+ \cdot [(\underset{\sim}{t} \cdot \bar{\underset{\sim}{F}})^+ - (\underset{\sim}{t} \cdot \bar{\underset{\sim}{F}})^-]\, ds \Big\}$$

$$= \underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t} \cdot \underset{\sim}{F})]\, ds \qquad (36)$$

$$\equiv \underline{J}.$$

Note that the limit $\varepsilon \to 0$ in the term within brackets { } in the extreme left hand side of Eq. (36) implies that: (a) in the volume integrals, a volume, however small, near the crack tip must be deleted, and (b) in the crack face integrals, an area, however small, near the crack-front must be deleted.

<u>Now, in view of Eq. (29) it is seen that the integral on the extreme left-hand-side of Eq. (36) is path-independent</u>* (Note however, the volume $V-V_\varepsilon$ change for each path). It is worth noting that Eq. (36) implies that the path 234, as in Fig. 2a, begins at the point 2 on the lower flank of the crack, and ends at point 4 on the upper flank, such that points 2 and 4 are equidistant from the crack-tip. However, this is just for convenience, points 2 and 4 <u>need not</u> be equidistant; in which case the integrals on Γ_{12} in Eq. (36) must be "split up" into integrals on both Γ_{12} and Γ_{45} as in Eq. (31).

Evidently, if one uses Eq. (27) and repeats the steps in Eqs. (31-36), we obtain an equivalent representation for a path-independent integral:

$$\int_{\Gamma_{234}} [\underline{N}(W-T) - \underline{N} \cdot \underset{\sim}{s} \cdot (2\gamma + I)]\, ds + \underset{\varepsilon \to 0}{\mathrm{Lt}} \Big\{ - \int_{V-V_\varepsilon} \rho_o \underline{f} \cdot \underset{\sim}{F} dv$$

* Note the fundamental difference in the notions of path-independence as embodied in Eqs. (30, 39) (which are due to [2,4]) on the one hand, and the present Eq. (36) on the other. In Eq. (36) it implies that the integral on the extreme L.H.S. evaluated on the contours Γ_{12345} or $\Gamma_{12'3'4'5}$ (see Fig. 2a) has the same value.

$$+ \int_{V-V_\varepsilon} \frac{d}{dt} (\rho_o \dot{\underset{\sim}{u}}.\underset{\sim}{F})\, dv + \int_{\Gamma_{12}} \underline{N}^+[(W^+-W^-) - (T^+-T^-)]\, ds - \int_{\Gamma'_{12}} [(\bar{\underset{\sim}{t}}.\underset{\sim}{F})^+$$

$$+ (\bar{\underset{\sim}{t}}.\underset{\sim}{F})^-]\, ds - \int_{\Gamma''_{12}} \underline{N}^+\cdot[(\underset{\sim}{s}.\underset{\sim}{F}^T.\bar{\underset{\sim}{F}})^+ - (\underset{\sim}{s}.\underset{\sim}{F}^T.\bar{\underset{\sim}{F}})^-]\, ds \Big\} \equiv \underline{J}. \qquad (37)$$

Trivial as the difference may be from a theoretical view point, Eq. (37) is in fact more convenient from a computational view point, to calculate $\underline{J}$ from a far-field contour, than Eq. (36), since most generally available computer programs (finite-element!) use $\underset{\sim}{s}$ and $\underset{\sim}{\gamma}$ as primitive variables [12].

If we consider the special case when (i) deformations are infinitesimal, and thus the distinction between various stress measures vanish, i.e., $\underset{\sim}{t} \approx \underset{\sim}{\sigma}$, (ii) the material is linear elastic and homogeneous, (iii) the material is under dynamic equilibrium, and (iv) the geometries of the solid and crack are conveniently described in a cartesian system, then Eq. (36) is reduced, at any time t, to:

$$J_K = \int_{\Gamma_{234}} [\, N_K(W-T) - N_M \sigma_{MJ} (\delta_{JK} + u_{J,K})]\, ds$$

$$+ \underset{\varepsilon\to 0}{\mathrm{Lt}} \Big\{ - \int_{V-V_\varepsilon} \rho_o f_J (\delta_{JK} + u_{J,K})\, dv + \int_{V-V_\varepsilon} \frac{d}{dt} [\rho_o \dot{u}_J (\delta_{JK} + u_{J,K})]\, dv \Big\}$$

$$+ \int_{\Gamma_{12}} N_J^+ [(W^+-W^-) - (T^+-T^-)]\, ds - \int_{\Gamma'_{12}} [\langle \bar{t}_J (\delta_{JK} + u_{J,K}) \rangle^+ + \langle \bar{t}_J (\delta_{JK} + u_{J,K}) \rangle^-]\, ds$$

$$- \int_{\Gamma''_{12}} N_J^+ [(t_{JL} \bar{F}_{LK})^+ - (t_{JL} \bar{F}_{LK})^-]\, ds. \qquad (38)$$

This should be contrasted with the expression for linear elasto-static case, when no conditions are prescribed on the crack-face, given in [2,4] denoted here for comparison purposes, as $\underline{J}^*$:

$$J_K^* = \int_{\Gamma_{234}} (W N_K - N_M \sigma_{MJ} u_{J,K}]\, ds. \qquad (39)$$

Thus, in the present $\underline{J}$, Eq. (38), discontinuities of W and T across the

crack-faces* are allowed for, in addition to its being applicable to the general case as discussed.

It is interesting to note however, that even in the general case represented by Eq. (36), the following identity holds:

$$- \int_{\Gamma_{234}} N_M t_{MJ} \delta_{JK} + \underset{\varepsilon \to 0}{\mathrm{Lt}} \Big\{ - \int_{V-V_\varepsilon} \rho_o f_J \delta_{JK}$$

$$+ \int_{V-V_\varepsilon} \frac{d}{dt} (\rho_o \dot{u}_J \delta_{JK})\, dv - \int_{\Gamma'_{12}} < (\bar{t}_J \delta_{JK})^+ + (\bar{t}_J \delta_{JK})^- > ds \Big\}$$

$$= \underset{\varepsilon \to 0}{\mathrm{Lt}} - \int_{\Gamma_\varepsilon} N_M t_{MJ} \delta_{JK} ds \tag{40}$$

since, Eq. (40) is nothing other than the global linear momentum balance condition for the domain $V-V_\varepsilon$ in the limit $\varepsilon \to 0$. Thus we may reduce Eq. (36), through a slight modification to the definition of $\underline{J}$, as:

$$\int_{\Gamma_{234}} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t} . \underset{\sim}{e})]\, ds + \underset{\varepsilon \to 0}{\mathrm{Lt}} \Big\{ - \int_{V-V_\varepsilon} \rho_o \underline{f} . \underset{\sim}{e} dv$$

$$+ \int_{V-V_\varepsilon} \frac{d}{dt} (\rho_o \underline{\dot{u}} . \underset{\sim}{e})\, dv + \int_{\Gamma_{12}} \underline{N}^+ [(W^+ - W^-) - (T^+ - T^-)]\, ds$$

$$- \int_{\Gamma'_{12}} [(\underline{\bar{t}} . \underset{\sim}{e})^+ + (\underline{\bar{t}} . \underset{\sim}{e})^-]\, ds - \int_{\Gamma''_{12}} \underline{N}^+ . [(\underset{\sim}{t} . \bar{\underset{\sim}{e}})^+ - (\underset{\sim}{t} . \bar{\underset{\sim}{e}})^-]\, ds \Big\}$$

$$= \underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [\underline{N}(W-T) - \underline{N} \cdot (\underset{\sim}{t} . \underset{\sim}{e})]\, ds$$

$$\equiv \underline{J}. \tag{41}$$

Thus Eq. (41) represents a slight modification to the definition of $\underline{J}$ in as much as $\underset{\sim}{F}$ in Eq. (36) is now replaced by $\underset{\sim}{e}[\underset{\sim}{e} = \underset{\sim}{F} - \underset{\sim}{I}]$.

We now examine the physical interpretation of $\underline{J}$ as presently defined through Eq. (41). For this purpose let us consider the volume V_ε at time t. It is to be understood that V_ε is a small region at the crack-tip (front) with

* Note that, based on physical considerations, we may assume that W^+ and W^- along the crack-face, in general, may have integrable singularities near the crack-tip (front). An example wherein $(W^+ - W^-) \neq 0$ at the crack-face has recently been brought to the author's attention [13].

the surface Γ_ε. Let ψ_ε be the potential of external forces acting on V_ε, and let W^*_ε and T^*_ε be the strain energy and kinetic energy, respectively of V_ε. It is seen that

$$\psi_\varepsilon = -\int_{V_\varepsilon} \bar{f}_i u_i dv - \int_{S_{t\varepsilon}} \bar{t}_i u_i ds - \int_{\Gamma_\varepsilon} \left\{ \int_0^{u_i} t_i(\mu_i) d\mu_i \right\} ds. \tag{42}$$

where $S_{t\varepsilon}$ is the crack surface(s) enclosed by Γ_ε. The last term on the r.h.s. of Eq. (42) is the work of tractions exerted on V_ε by the surrounding solid, and these tractions are dependent on the displacement field. Likewise, we have:

$$W^*_\varepsilon = \int_{V_\varepsilon} W dv = \int_{V_\varepsilon} \left\{ \int_0^{F_{ij}} t_{mn}(\mu_{ij}) d\mu_{nm} \right\} dv \tag{43}$$

and

$$T^*_\varepsilon = \int_{V_\varepsilon} T dv = \int_{V_\varepsilon} \frac{1}{2} \rho \dot{u}_i \dot{u}_i \, dV \tag{44}$$

where $\dot{u}_i$ is the <u>absolute</u> velocity of a material particle. In Eqs. (42-44) cartesian coordinates x_k have been used for simplicity. Let c_k be the cartesian coordinates of the crack-tip. Let $\zeta_k (=x_k - c_k)$ be coordinates centered at the crack-tip when the crack is of "length" c_k. It is seen that in general, in the immediate vicinity of the crack-tip, i.e., in V_ε, which may be considered to be a "process-zone", we may assume:

$$u_i = u_i(\zeta_k, c_k, \dot{c}_k, t); \; t_{ij} = t_{ij}(\zeta_k, c_k, \dot{c}_k, t)$$

$$W = W(\zeta_k, c_k, \dot{c}_k, t); T = T(\zeta_k, c_k, \dot{c}_k, t). \tag{45}$$

Thus the variables u_i, $\underset{\sim}{t}$, W, and T may depend explicitly on the crack "length" c_k, crack-velocity $\dot{c}_k$, as well as time t. It is known that in the vicinity of a sharp crack-tip (front), the displacements u_i are nonsingular (even though the material velocity $\dot{u}_i$ may be singular), while W and T are singular; and based on physical considerations, the singularities in W and T can be of order

r^{-1} where r, θ are polar coordinates centered at the crack-tip. Even in finite bodies the representations (45) may be considered as being asymptotically correct within the "process-zone" irrespective of the geometry of the body and loading. We may write at time t, for instance,

$$W^* = \int_{V_\varepsilon} W(\zeta_K, c_K, \dot{c}_k, t)\ dv \tag{46}$$

and a similar expression for T* of v_ε.

Now, let us assume that in time (dt) the crack "advances" by dc_k. Let ζ_k' be once again, the cartesian coordinates centered at the new crack-tip such that $\zeta_k'=\zeta_k-dc_k$ as shown in Fig. 3. Also, without loss of generality, let $\varepsilon=\beta dc_k$, $\beta>1$, such that the crack-tip which is within V_ε (say a circle of radius ε centered at the crack-tip) at time t, will not lie outside the fixed V_ε at time (t+dt). Thus we are considering the case when the crack-tip advances by "dc_k" in time 'dt' <u>into</u> a fixed volume V_ε. Analogous to Eq. (46), we may then write:

$$W^*(t+dt) = \int_{V_\varepsilon} W(\zeta_k', c_k+dc_k, \dot{c}_k+d\dot{c}_k, t+dt)\ dv. \tag{47}$$

We note that, in the asymptotic sense, the dependence of W on ζ_k' at t+dt for a sharp crack is, in general, of the same functional form as the dependence of W on ζ_k at t. Note also that in performing the integration as in Eq. (47), the limits of integration for a <u>fixed</u> volume V_ε in terms of ζ_k' would, of course, be different from those in terms of ζ_k in Eq. (46) (see Figs. 3a and 3b). However, by noting that $\zeta_k'=\zeta_k-dc_k$, and $d\zeta_k'=d\zeta_k$, we have, by what amounts to a change of variables,

$$W^*(t+dt) = \int_{V_\varepsilon} W[(\zeta_k-dc_k),(c_k+dc_k),(\dot{c}_k+d\dot{c}_k),(t+dt)]\ dv. \tag{48}$$

Now the limits of integration in Eq. (48), would be the same as those in Eq.

(47). Now, we may expand out the integrand on r.h.s. of Eq. (48) using Taylor series and express it in terms of $W(\zeta_k, c_k, \dot{c}_k, t)$. However, it is recalled that W is singular (viz., r^{-1} for a sharp crack) w.r.t. ζ_k as $\zeta_k \to 0$, where as the explicit dependence of W on c_k, $\dot{c}_k$ and t is in general, non-singular. Thus, while $\partial W/\partial c_k$, $\partial W/\partial \dot{c}_k$, and $\partial W/\partial t$ are all integrable, the partial derivative $\partial W/\partial \zeta_k$ would, however, be non-integrable. Thus the above idea of <u>changing variables</u> is non-workable. Thus, we rewrite Eq. (47) as:

$$W^*_\varepsilon(t+dt) = \int_{V_\varepsilon} [W(\zeta_k', c_k, \dot{c}_k, t) + \frac{\partial W}{\partial c_k} dc_k + \frac{\partial W}{\partial \dot{c}_k} d\dot{c}_k + \frac{\partial W}{\partial t} dt]\, dv. \qquad (48)$$

Now consider the term:

$$I = \int_{V_\varepsilon} W(\zeta_k', c_k, \dot{c}_k, t)\, dv - \int_{V_\varepsilon} W(\zeta_k, c_k, \dot{c}_k, t) dv. \qquad (49)$$

Since the functional dependence of W on ζ_k' and ζ_k are of the same form, we can "subtract out" the singularities in evaluating I of (49) as shown in Fig. 3c. From Fig. 3c it is apparent that the term I of (49) is given by:

$$I = - \int_{\Gamma_\varepsilon} (W N_k dc_k)\, ds. \qquad (50)$$

Note the negative sign on the r.h.s. of Eq. (50) is due to the definition of the "outward" normal to the contour Γ in the sense indicated in Fig. 3c. [i..e, a contour beginning at the "bottom" surface of crack and ending on the "top" surface]. Let us now define the derivative $[D(\ )/Dt]^c$ as the total rate of change of ( ) in a time "dt" <u>due to</u> crack growth by "dc_k". Thus, for instance,

$$[\frac{DW^*_\varepsilon}{Dt}]^c dt = W^*_\varepsilon(t+dt) - W^*_\varepsilon(t). \qquad (51)$$

From Eqs. (48), (49) and (50) it is seen that:

$$[\frac{DW^*_\varepsilon}{Dt}]^c = - \int_{\Gamma_\varepsilon} W N_k \dot{c}_k ds + \int_{V_\varepsilon} (\frac{\partial W}{\partial c_k} \dot{c}_k + \frac{\partial W}{\partial \dot{c}_k} \ddot{c}_k + \frac{\partial W}{\partial t})\, dv. \qquad (52)$$

In the case of a dynamically propagating crack, the kinetic energy T may also have (as is known [8], for instance, in linear elastodynamics) a singularity. Thus, following essentially the same arguments as in Eqs. (46-50), we have:

$$[\frac{DT^*_\varepsilon}{Dt}]^c = -\int_{\Gamma_\varepsilon} TN_k\dot{c}_k ds + \int_{V_\varepsilon} (\frac{\partial T}{\partial c_k}\dot{c}_k + \frac{\partial T}{\partial \dot{c}_k}\ddot{c}_k + \frac{\partial T}{\partial t})\, dv. \tag{53}$$

Finally, at times t and (t+dt), respectively, the displacements are given by u_i and $[u_i + (\frac{Du_i}{Dt})^c dt]$. From the Equations:

$$u_i[t] = u_i(\zeta_k, c_k, \dot{c}_k, t)$$

and

$$u_i[t+dt] = u_i(\zeta_k', c_k+dc_k, \dot{c}_k+d\dot{c}_k, t+dt)$$

$$= u_i(\zeta_k-dc_k, c_k+dc_k, \dot{x}_k+d\dot{c}_k, t+dt) \tag{54}$$

it is seen that:

$$(\frac{Du_i}{Dt})^c = -\frac{\partial u_i}{\partial \zeta_k}\dot{c}_k + \frac{\partial u_i}{\partial c_k}\dot{c}_k + \frac{\partial u_i}{\partial \dot{c}_k}\ddot{c}_k + \frac{\partial u_i}{\partial t}. \tag{5'}$$

Using Eq. (55), it is then seen from (42) that:

$$(\frac{D\psi_\varepsilon}{Dt})^c = -\int_{V_\varepsilon} \bar{f}_i (\frac{Du_i}{Dt})^c dv - \int_{S_{t\varepsilon}} \bar{t}_i (\frac{Du_i}{Dt})^c ds - \int_{\Gamma_\varepsilon} t_i (\frac{Du_i}{Dt})^c ds \tag{.6)}$$

where t_i are tractions corresponding to u_i at Γ_ε at t. We will now consider the "energy release rate" to the crack-tip, as measured in the process-zone, V_ε. This energy release rate, denoted here by $(DE_\varepsilon/Dt)^c$, is given, from the energy balance within V_ε, by:

$$(\frac{DE_\varepsilon}{Dt})^c = -(\frac{D\psi_\varepsilon}{Dt})^c - [(\frac{DW_\varepsilon}{Dt})^c + (\frac{DT_\varepsilon}{Dt})]\,. \tag{57}$$

Upon using (52), (53) and (56), we write (57) as:

$$(\frac{DE}{Dt})^c = \dot{c}_k \left\{ \int_{\Gamma_\varepsilon} [(W+T)N_k - t_i \frac{\partial u_i}{\partial \zeta_k}]\, ds - \int_{V_\varepsilon} \bar{f}_i \frac{\partial u_i}{\partial \zeta_k} dv - \int_{S_{t\varepsilon}} \bar{t}_i \frac{\partial u_i}{\partial \zeta_k}\, ds \right\}$$

$$- \dot{c}_k \left\{ \int_{V_\varepsilon} \left(\frac{\partial W}{\partial c_k} + \frac{\partial T}{\partial c_k} - \bar{f}_i \frac{\partial u_i}{\partial c_k} \right) dv - \int_{\Gamma_\varepsilon} t_i \frac{\partial u_i}{\partial c_k} ds - \int_{S_{t\varepsilon}} \frac{\partial u_i}{\partial c_k} ds \right\}$$

$$- \ddot{c}_k \left\{ \int_{V_\varepsilon} \left(\frac{\partial W}{\partial \dot{c}_k} + \frac{\partial T}{\partial \dot{c}_k} - \bar{f}_i \frac{\partial u_i}{\partial \dot{c}_k} \right) dv - \int_{\Gamma_\varepsilon} t_i \frac{\partial u_i}{\partial \dot{c}_k} ds - \int_{S_{t\varepsilon}} \bar{t}_i \frac{\partial u_i}{\partial \dot{c}_k} ds \right\}$$

$$- \left\{ \int_{V_\varepsilon} \left(\frac{\partial W}{\partial t} + \frac{\partial T}{\partial t} - \bar{f}_i \frac{\partial u_i}{\partial t} \right) dv - \int_{\Gamma_\varepsilon} t_i \frac{\partial u_i}{\partial t} ds - \int_{S_{t\varepsilon}} \frac{\partial u_i}{\partial t} ds \right\}$$

$$\equiv (I) + (II) + (III) + (IV) \tag{58}$$

where (I), (II), (III) and (IV) identify, respectively the first through fourth terms each enclosed within { }, on the r.h.s. of Eq. (58). Suppose now that for a dynamically propagating crack, W, T, and $t_{ij}\ \partial u_j/\partial x_i$ [note that $\partial(\ )/\partial x_i \equiv \partial(\ )/\partial \zeta_i$] all have singularities of the type r^{-1}. On the other hand, the explicit dependence of these quantities on c_k, $\dot{c}_k$, and t are, in general, such that their partial derivatives w.r.t. c_k, $\dot{c}_k$, and t are also singular, but the singularity is still of order r^{-1}. Suppose we consider a 2-dimensional problem wherein V_ε is a circular domain of radius $\varepsilon(\equiv\beta dc_k)$ centered at the crack tip;

$$\int_{\Gamma_\varepsilon} (\qquad) ds = \int_{-\Pi}^{+\Pi} (\qquad) \varepsilon d\theta \tag{59a}$$

$$\int_{V_\varepsilon} (\qquad) dv = \int_0^\varepsilon \int_{-\Pi}^{+\Pi} (\qquad) r dr d\theta . \tag{59b}$$

Thus, for a 2-dimensional problem (the same argument carries over to 3-dimensional case) for instance, the term (I) of Eq. (58) becomes:

$$(I) = \dot{c}_k \left\{ \int_{-\Pi}^{+\Pi} \left[(W+T)N_k - t_i \frac{\partial u_i}{\partial \zeta_k} \right]_{(r=\varepsilon)} \varepsilon d\theta - \int_0^\varepsilon \int_{-\Pi}^{+\Pi} \bar{f}_i \frac{\partial u_i}{\partial \zeta_k} r dr d\theta \right.$$

$$\left. - \int_0^\varepsilon \left[\left(\bar{t}_i \frac{u_i}{\partial \zeta_k} \right)_{-\Pi} + \left(\bar{t}_i \frac{\partial u_i}{\partial \zeta_k} \right)_{+\Pi} \right] dr \right\} \tag{60}$$

where $S_{t\varepsilon}$ is defined by $\theta = \pm\Pi$. Supposing that $\bar{f}_i$ and $\bar{t}_i$ are non-singular

near the crack-tip, while W, T and $t_i u_{i,k}$ behave as r^{-1}, it is seen that the first term on the r.h.s. of (60) is <u>independent</u> of ε, while the other terms are $O(\varepsilon)$. Likewise, it is observed that terms (II), (III) and (IV) of Eq. (58a) are all of $O(\varepsilon)$. Thus, in general, we have from (58),

$$\left(\frac{DE_\varepsilon}{Dt}\right)^c = \dot{c}_k \left\{ \int_{\Gamma_\varepsilon} \left[(W+T)N_k - t_i \frac{\partial u_i}{\partial \zeta_k} \right] ds \right\} + O(\varepsilon). \tag{61}$$

Hence, in the limit, we may take for an arbitrarily small volume near the crack-tip,

$$\left(\frac{DE_\varepsilon}{Dt}\right)^c = \dot{c}_k \left\{ \int_{\Gamma_\varepsilon} \left[(W+T)N_k - t_i \frac{\partial u_i}{\partial \zeta_k} \right] ds \right\}. \tag{62}$$

$$\equiv \dot{c}_k \int_{\Gamma_\varepsilon} \left[(W+T)N_k - t_i \frac{\partial u_i}{\partial x_k} \right] ds. \tag{62a}$$

Comparing Eqs. (62a) and (41) it is seen that:

$$\left(\frac{DE_\varepsilon}{Dt}\right)^c = \dot{c}_k \left\{ J_k + 2 \int_{\Gamma_\varepsilon} TN_k ds \right\}. \tag{63}$$

Thus, in the case of <u>propagating cracks</u> in elastodynamic fields, the path independent integral J_k of Eqs. (41), and hence Eq. (38), <u>does not</u> have the physical meaning of energy release rate per unit crack-growth. However, if we define the "Lagrangean" of the domain V as:

$$L_\varepsilon = - \psi_\varepsilon - W^*_\varepsilon + T^*_\varepsilon \tag{64}$$

it can be immediately seen from Eq. (57), and the development that follows thereon, that:

$$\left(\frac{DL_\varepsilon}{Dt}\right)^c = \left(\frac{DE_\varepsilon}{Dt}\right)^c + 2\left(\frac{DT^*_\varepsilon}{Dt}\right)^c$$

$$= \dot{c}_k \int_{\Gamma_\varepsilon} \left[(W-T)N_k - t_i \frac{\partial u_i}{\partial x_k} \right] ds$$

$$= \dot{c}_k J_k . \tag{64b}$$

Thus the path-independent integral of (41), and hence (38), while not representing an "energy release", does still have a physically meaningful interpretation of rate change of Lagrangean per unit crack growth.

On the other hand, for stationary cracks in dynamic elastic fields T is nonsingular, while, in elasto-statics T is negligible altogether. In these cases, it can be seen that:

$$\left(\frac{DE_\varepsilon}{Dt}\right)^c dt = dc_k J_k \tag{65}$$

and thus, J_k does have the physical meaning that it is the energy-release rate per unit movement of the crack-tip in x_k direction.

In the linear-elastic case, an expression for energy-release rate for a crack extending in an elasto-dynamic field was derived by Freund [7] to be:

$$F = \lim_{s\to 0} \int_s \left(\sigma_{ij} n_j \dot{U}_i + \frac{1}{2}\sigma_{ij} U_{i,j} v_n + \frac{1}{2}\rho \dot{U}_i \dot{U}_i v_n\right) ds. \tag{66}$$

where, in the notation of [7]: s is a "small" loop that moves along with the crack-tip; σ_{ij} is the stress tensor; U_i are displacements; $\dot{U}_i$ are velocities, n_j are direction cosines of a unit normal to s pointing away from the crack-tip, and v_n is the "component of velocity of a point on s in the direction of n_i (if the crack-tip is moving in the x_1 - direction with instantaneous speed v, then $v_n = vn_1$). In deriving the above expression (66), a small loop of unchanging geometry was supposed to be moving with the crack-tip and a global energy rate balance was employed [7].

It is seen that the presently derived expression for energy release rate, viz., Eq. (62a) differs from that derived in [7], viz., (66). As demonstrated earlier, the present Eq. (62a) can be readily reduced to the well-known results for energy release rates for elasto-static crack problem as well as for the elasto-dynamic problem when the crack is stationary. On the other hand Eq. (66) of [7] is not readily reducible to the above cases, without invoking

certain other assumptions.

We now consider finite-strain inelasticity, but restrict our attention to the two cases: (i) finite strain, classical rate-independent elastoplasticity, and (ii) finite strain rate-dependent elasto-viscoplasticity including creep as a special case.

Finite-Strain, Rate Independent, Classical Elasto-Plasticity:

Because of the author's own interest in computational mechanics, the following is presented in a fashion that is directly amenable to computations based on, say, the finite element method. However, the basic development itself is divorced from any "computational" overtones, lest they may be deemed "bad".

At time t, the material particle P of B is located at p in b. At time $t + \Delta t$, let the same material particle move to p_1. As discussed before, the spatial coordinates in b are η^m, and the convected coordinates are ξ^J. Let the vector pp_1 be $\Delta \underline{u}$.

$$\Delta \underline{u} = \Delta u^J \underline{G}_J = \Delta u^m \underline{g}_m = \Delta u^{J*} \underline{g}_J . \tag{67}$$

We define the incremental-displacement-gradient tensor $\Delta \underset{\sim}{e}$ as:

$$\begin{aligned} \Delta \underset{\sim}{e} &= \Delta u_{m;n} \underline{g}^m \underline{g}^n \\ &= (\Delta \varepsilon_{mn} + \Delta \omega_{mn}) \; \underline{g}^m \underline{g}^n \end{aligned} \tag{68}$$

where $\Delta \underset{\sim}{\varepsilon}$ and $\Delta \underset{\sim}{\omega}$ are, respectively, the incremental strain and incremental spin referred to the current configuration at t and,

$$\Delta \underset{\sim}{\varepsilon} = \frac{1}{2}(\Delta u_{m;n} + \Delta u_{n;m}) \; \underline{g}^n \underline{g}^m ; \; \Delta \underset{\sim}{\omega} = \frac{1}{2} \; (\Delta u_{m;n} - \Delta u_{n;m}) \; \underline{g}^m \underline{g}^n . \tag{69}$$

In the above and in the preceding, $(\;)_{;n}$ and $(\;)_{;J}$ refer to covariant derivatives (w.r.t. η^n and ξ^J respectively at p, using the metric g_{mn}, and g_{JK} respectively) in the current configuration. Let the Cauchy stress and Kirchhoff stresses, $\underset{\sim}{\tau}$ and $\underset{\sim}{\sigma}$ respectively, at time t be represented as:

$$\underset{\sim}{\tau} = \tau^{mn}\underline{g}_m\underline{g}_n = \tau^{J*K*}\underline{g}_J\underline{g}_K$$

$$\underset{\sim}{\sigma} = J\underset{\sim}{\tau} = J\underset{\sim}{\tau}^{mn}\underline{g}_m\underline{g}_n = J\tau^{J*K*}\underline{g}_J\underline{g}_K. \tag{70}$$

Let $\Delta\underset{\sim}{\sigma}$ be the total, or substantial, or material increment of Kirchhoff stress, such that,

$$\Delta\underset{\sim}{\sigma} = \dot{\underset{\sim}{\sigma}}\Delta t = \left(\frac{D\sigma^{mn}}{Dt}\right)\Delta t\underline{g}'_m\underline{g}_n \tag{71}$$

where $\dot{\underset{\sim}{\sigma}}$ is the material derivative (for a fixed material particle, $\xi^J =$ const.), of $\underset{\sim}{\sigma}$. For an objective stress-rate, to be used in the incremental constitutive relation for an elastic-plastic solid, we take, following Hill [14], the co-rotational increment of $\underset{\sim}{\sigma}$ (which is often referred to as the Zaremba, or the Jaumann, or the rigid-body rate), which is denoted here by $\Delta\hat{\underset{\sim}{\sigma}}$. It is well-known (see for instance [14]) that

$$\Delta\hat{\underset{\sim}{\sigma}} = \Delta\underset{\sim}{\sigma} + \underset{\sim}{\sigma} \cdot \Delta\underset{\sim}{\omega} - \Delta\underset{\sim}{\omega} \cdot \underset{\sim}{\sigma}. \tag{72}$$

A constitutive law for *the rate-theory of plasticity*, as suggested by Hill [14] is:

$$\Delta\hat{\underset{\sim}{\sigma}} = \frac{\partial\Delta V}{\partial\Delta\underset{\sim}{\varepsilon}} \tag{73}$$

where

$$\Delta V = \frac{1}{2} L_{mnpq}\Delta\varepsilon^{mn}\Delta\varepsilon^{pq} - \left(\frac{\alpha}{g}\right)(\lambda_{k\ell}\Delta\varepsilon^{k\ell})^2. \tag{74}$$

Eq. (74) leads to a bilinear relation. In Eq. (74): L_{mnpq} is a tensor of instantaneous elastic modulii, which is + ve definite and symmetric under $mn \leftrightarrow pq$ interchange; $\alpha=1$, or 0 according to whether $\underset{\sim}{\lambda}:\Delta\underset{\sim}{\varepsilon}$ is + ve or - ve, λ_{mn} is a tensor normal to the interface between elastic and plastic domains in the strain space, and g is a scalar related to a measure of rate of hardening due to plastic deformation. For classical isotropically hardening materials, the above constitutive law, which has been used by several authors in the past few years, as discussed in [15], becomes:

$$\Delta\underset{\sim}{\sigma} = 2\mu\Delta\underset{\sim}{\varepsilon} + \lambda(\Delta\underset{\sim}{\varepsilon}:\underset{\sim}{I})\underset{\sim}{I} - \frac{12\alpha\mu^2(\Delta\underset{\sim}{\varepsilon}:\underset{\sim}{\sigma}')\underset{\sim}{\sigma}'}{(\underset{\sim}{\sigma}':\underset{\sim}{\sigma}')[6\mu+2(\partial F/\partial W^P)]} \tag{75}$$

where λ and μ are Lame's constants; $\underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - \frac{1}{3}(\underset{\sim}{\sigma}:\underset{\sim}{I})\underset{\sim}{I}$ is the deviatoric Kirchhoff stress at p, and the yield-surface is represented by: $F = [3J_2(\underset{\sim}{\sigma}')]^{\frac{1}{2}} - F_o(W^P) = 0$; $W^P = \int \underset{\sim}{\sigma}:\Delta\underset{\sim}{\varepsilon}^P dt$ and $J_2(\underset{\sim}{\sigma}') = (1/2)(\underset{\sim}{\sigma}':\underset{\sim}{\sigma}')$.

For purposes of the ensuing discussion, it should be noted that the constant α (which is equal to 1 or 0) in Eqs. (74, 75) is a function of the spatial coordinates η^m in b. Thus, a generic point p in b may be experiencing loading (α=1) or unloading (α=0). Thus at time t, α depends on the location of p in b.

Let $(\underset{\sim}{\tau}+\Delta\underset{\sim}{t})$ represent the first Piola-Kirchhoff stress at p_1 at time $(t+\Delta t)$, as referred to the configuration b at time t. Then it is seen [15] that:

$$\Delta\underset{\sim}{t} = (\Delta\underset{\sim}{\sigma}-\Delta\underset{\sim}{e}.\underset{\sim}{\sigma})/J \tag{76}$$

$$= (\Delta\hat{\underset{\sim}{\sigma}}-\Delta\underset{\sim}{\varepsilon}.\underset{\sim}{\sigma}-\underset{\sim}{\sigma}.\Delta\underset{\sim}{\omega})/J \tag{77}$$

where $J=(\rho_o/\rho_t)$ the ratio of mass densities in B and b respectively. Use has been made of Eq. (72) in obtaining (77) from (76). Hence, in view of Eq. (73) it is seen [15] that a potential ΔJ exists such that:

$$\frac{\partial\Delta U}{\partial\Delta\underset{\sim}{e}^T} = \Delta\underset{\sim}{t} \tag{78}$$

and $$\Delta U = \Delta V - \underset{\sim}{\sigma}:(\Delta\underset{\sim}{\varepsilon}.\Delta\underset{\sim}{\varepsilon}) + \frac{1}{2}\underset{\sim}{\sigma}:(\Delta\underset{\sim}{e}^T.\Delta\underset{\sim}{e}) \tag{79}$$

Note $\Delta U = \Delta U[\alpha(p)]$, i.e., ΔU is a function of α(=0 or 1) which is in turn a function of p.

Let $\underline{a}$ be the absolute acceleration (i.e., the material derivative of the velocity vector), of the material particle at $(t+\Delta t)$. The acceleration $\underline{a}$ can be expressed in terms of spatial bases $\underset{\sim}{g}_m$ at t, or sometimes more conveniently (from a computational viewpoint) in the fixed bases $\underset{\sim}{G}_J$ at P at t=0. (see, for instance, [15].) The field equations at $t+\Delta t$ can now be written as:

$$\text{(LMB)}\quad \underline{\nabla}_t \cdot [\underset{\sim}{\tau}+\Delta\underset{\sim}{t}] + \rho_t\,(\underline{f}-\underline{a}) = 0. \tag{80}$$

$$\text{(AMB)}\quad \Delta\underset{\sim}{e} \cdot \underset{\sim}{\tau} + \Delta\underset{\sim}{t} = \Delta\underset{\sim}{t}^T + \underset{\sim}{\tau} \cdot \Delta\underset{\sim}{e}^T \tag{81}$$

$$\Delta\underset{\sim}{t} = \partial\Delta U/\partial\Delta\underset{\sim}{e}^T;\quad \Delta\underset{\sim}{e} = \Delta u_{m;n}\underline{g}^m\underline{g}^n \tag{82,83}$$

$$\underline{n}_t \cdot [\underset{\sim}{\tau}+\Delta\underset{\sim}{t}] = \bar{\underline{t}} \text{ at } S_t \tag{84}$$

$$\text{and}\quad \Delta\underset{\sim}{e} = \Delta\bar{\underset{\sim}{e}} \text{ at } S_e. \tag{85}$$

Note that $\underline{\nabla}_t = \underline{g}^m(\partial/\partial\eta^m)$ is the gradient operator at p at time t. In the above, $\bar{\underline{t}}$ are prescribed tractions at $(t+\Delta t)$ as measured per unit area of surface S_t at time t. It should be noted that when the potential ΔU for $\Delta\underset{\sim}{t}$ is as defined in Eq. (79), the AMB Eq. (81) is automatically embedded in the structure of ΔU, as discussed in [15].

Now we show that for a closed volume v_t (at time t) wherein α=constant (i.e., a volume which is everywhere experiencing loading or everywhere experiencing unloading), and which is free from any singularities or other defects, the following vector integral is zero.

$$\underline{0} = \int_{v_t}\Big\{\underline{\nabla}_t(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U) - (\underline{\nabla}_t\tau) : \Delta\underset{\sim}{e} - \underline{\nabla}_t \cdot [(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot \Delta\underset{\sim}{e}]$$

$$- \rho_t(\underline{f}-\underline{a}) \cdot \Delta\underset{\sim}{e}\Big\}\,dv + \int_{S_t}[\underline{n}_t \cdot (\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) - \bar{\underline{t}}] \cdot \Delta\underset{\sim}{e}ds$$

$$+ \int_{S_e}\underline{n} \cdot (\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot (\Delta\underset{\sim}{e}-\Delta\bar{\underset{\sim}{e}})\,ds. \tag{86}$$

Note again that, in general, $\Delta U = \Delta U[\alpha(p)]$ but, within the considered v_t, α = constant = 1 or = 0. The proof of the above statement is based on the results:

$$\underline{\nabla}_t[\underset{\sim}{\tau}:\Delta\underset{\sim}{e}] = (\underline{\nabla}_t\underset{\sim}{\tau}) : \Delta\underset{\sim}{e} + [\underset{\sim}{\tau} : (\Delta\underset{\sim}{e})_{;m}]\,\underline{g}^m \tag{87}$$

$$\underline{\nabla}_t\Delta U = [\Delta\underset{\sim}{t}^T : (\Delta\underset{\sim}{e})_{;m}]\,\underline{g}^m. \tag{88}$$

Since,

$$\underset{\sim t}{\nabla} \alpha(p) = 0 \text{ for considered } v_t \tag{89}$$

$$\underset{\sim t}{\nabla} \cdot [(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot \Delta\underset{\sim}{e}] = [\underset{\sim t}{\nabla} \cdot (\underset{\sim}{\tau}+\Delta\underset{\sim}{t})] \cdot \Delta\underset{\sim}{e} + [(\underset{\sim}{\tau}+\Delta\underset{\sim}{t})^T : (\Delta\underset{\sim}{e});_m]\, \underset{\sim}{g}^m . \tag{90}$$

From Eqs. (87-89), (80), and (82-85), the validity of Eq. (86) is immediately evident.

For instance, in a 2-dimensional dynamic elasto-plastic problem, consider two paths* Γ_1^* and Γ_2^* surrounding the crack-tip as shown in Fig. 1. We note that within the region $(v_1)_t$ enclosed by Γ_1^*, there can be regions of both plastic loading and elastic unloading, i.e., α can be either 1 or 0 at various points within $(v_1)_t$. However, within the region $(v_2-v_1)_t$, i.e., the region between the paths Γ_1^* and Γ_2^*, there can be either only loading or only unloading, i.e., within the region $(v_2-v_1)_t$, either $\alpha=1$ only or $\alpha=0$ only. In this case, we have in view of Eq. (86):

$$\underset{\sim}{0} = \int_{(v_2-v_1)_t} \Big\{ \underset{\sim}{\nabla}(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U) - (\underset{\sim t}{\nabla}\underset{\sim}{\tau}) : \Delta\underset{\sim}{e} - \underset{\sim t}{\nabla} \cdot [(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot \Delta\underset{\sim}{e}]$$

$$- \rho_t(\underset{\sim}{f}-\underset{\sim}{a})\cdot\Delta\underset{\sim}{e} \Big\} dv + \int_{(S_t)_1^2} [\underset{\sim t}{n}\cdot(\underset{\sim}{\tau}+\Delta\underset{\sim}{t})-\bar{\underset{\sim}{t}}]\cdot\Delta\underset{\sim}{e}\, ds + \int_{(S_e)_1^2} \underset{\sim t}{n}\cdot(\underset{\sim}{\tau}+\Delta\underset{\sim}{t})\cdot(\Delta\underset{\sim}{e}-\Delta\bar{\underset{\sim}{e}})ds \tag{91}$$

Recall that in the case of an elastic material, the basic result of vanishing of a certain integral in a volume free from singularities, i.e., Eq. (26), was applied to a region $V-V_\varepsilon$, and the integral vector $\underset{\sim}{J}$ was defined as the non-vanishing limit of a contour integral on Γ_ε in the limit $\varepsilon\to0$, as in Eqs. (36) or (41). Note that Eq. (91) of the elasto-plastic case *is entirely analogous* to Eq. (26) of the elastic-case. However, in order for Eq. (91) to apply to the region $V-V_\varepsilon$ in the limit $\varepsilon\to0$, the entire region ($V-V_\varepsilon$ limit $\varepsilon\to0$) must undergo either plastic loading only, or elastic unloading (or remain purely elastic) only. Such a situation is, in general, unlikely in the case of (growing) cracks in ductile materials, for a general domain $V-V_\varepsilon$. If indeed these conditions

*For convenience, Γ_{AFE} and Γ_{BCD} of Fig. 1 are now renamed Γ_1^* and Γ_2^*, respectively

are met, and hence Eq. (91) is applicable, in the domain $V-V_\varepsilon$ (Lt $\varepsilon \to 0$), then results entirely analogous to Eq. (36) or (41) can be derived, and a vector which is the limiting value of an integral on Γ_ε in the limit $\varepsilon \to 0$ can be identified. However, in general, let us consider the case when the conditions for applicability of Eq. (91) are not met in $V-V_\varepsilon$ (Lt $\varepsilon \to 0$) i.e., $\underline{\nabla}_t \alpha(p) \neq 0$ in $V-V_\varepsilon$. Referring to Fig. 2, it is seen, in the limit $\varepsilon \to 0$,

$$\underset{\varepsilon \to 0}{\mathrm{Lt}} \left[\int_{V-V_\varepsilon} \left\{ \underline{\nabla}_t (\tau : \Delta e + \Delta U) - (\underline{\nabla}_t \tau) : \Delta e - \underline{\nabla}_t \cdot [(\tau + \Delta t) \cdot \Delta e] \right. \right.$$

$$\left. - \rho_t (\underline{f} - \underline{a}) \cdot \Delta e \right\} dv + \int_{(S_t)^2_\varepsilon} [\underline{n}_t \cdot (\tau + \Delta t) - \underline{\bar{t}}] \cdot \Delta e ds$$

$$\left. + \int_{(S_e)^2_\varepsilon} \underline{n} \cdot (\tau + \Delta t) \cdot (\Delta e - \Delta \bar{e}) ds \right] \neq 0 = \underline{R} \text{ (say)} \qquad (92)$$

where $\underline{R}$ can be seen to be an integral over $V-V_\varepsilon$ of terms involving $\underline{\nabla}_t \alpha(p)$. *Using the divergence theorem we can rewrite Eq. (92), for instance, for a 2-dimensional domain indicated in Fig. 2a,*

$$\int_{\Gamma_{234}} [\underline{n} (\tau : \Delta e + \Delta U) - \underline{n} \cdot \langle (\tau + \Delta t) \cdot \Delta e \rangle] ds + \underset{\varepsilon \to 0}{\mathrm{Lt}} \left\{ \int_{V-V_\varepsilon} [-\underline{\nabla}_t \tau : \Delta e \right.$$

$$- \rho_t (\underline{f} - \underline{a}) \cdot \Delta e] dv + \int_{\Gamma_{12}} \underline{n}^+ [(\tau : \Delta e + \Delta U)^+ - (\tau : \Delta e + \Delta U)^-] ds$$

$$\left. - \int_{\Gamma'_{12}} [(\underline{\bar{t}} . \Delta e)^+ + (\underline{\bar{t}} . \Delta e)^-] ds - \int_{\Gamma''_{12}} \underline{n}^+ [\langle (\tau + \Delta t) \cdot \Delta \bar{e} \rangle^+ - \langle (\tau + \Delta t) . \Delta \bar{e} \rangle^-] ds \right\}$$

$$= \underline{R} + \underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [\underline{n} (\tau : \Delta e + \Delta U) - \underline{n} \cdot \langle (\tau + \Delta t) \cdot \Delta e \rangle] ds.$$

$$\equiv \Delta \underline{T} \text{ (say)}. \qquad (93)$$

It is noted that comments, concerning Γ_{12}, Γ'_{12}, Γ''_{12} etc., essentially similar to those made in connection with Eq. (36) apply in the case of (93) as well.

In view of Eq. (91) it is seen that the integral on the extreme L.H.S.

of Eq. (93) is "path-independent" (however the enclosed volumes $V-V_\varepsilon$ will be different) for any two paths Γ_{234} and $\Gamma_{2'3'4'}$ between which there is entirely loading or entirely unloading i.e., $\alpha=1$ or $\alpha=0$ everywhere between the two paths. Specifically, consider the situation when: (i) Γ_{234} encloses a volume V_1 (including the crack-tip) where there may be both loading and unloading taking place and (ii) $\Gamma_{2'3'4'}$ encloses a volume v_2 such that in v_2-v_1 also, both loading and unloading are occuring. Then the vector integrals $(\Delta \underline{T})_1$ (involving Γ_{234} on the extreme L.H.S. of Eq. (93)) and $(\Delta \underline{T})_2$ (involving $\Gamma_{2'3'4'}$) will not be equal. On the other hand, in the situation when (i) Γ_{234} encloses v_1 wherein there may be both loading and unloading occuring and (ii) $\Gamma_{2'3'4'}$ encloses v_2 such that in (v_2-v_1), either only loading or only unloading (or remain elastic) is taking place, the integrals $\Delta \underline{T}_1$ and $\Delta \underline{T}_2$ are equal:

It is noted that $\Delta \underline{T}$ integral as evaluated on Γ_{234} from Eq. (93) "measures" not only the severity of conditions near the crack-tip, but also the effect of transition from plastic to elastic zones in $V-V_\varepsilon$ enclosed by Γ_{234}.

Now we shall consider the physical meaning of the $\Delta \underline{T}$ integral introduced in Eq. (93). First, we note by the definition of the potential ΔU,

$$\underset{\sim}{\tau} : \Delta \underset{\sim}{e} + \Delta U = \underset{\sim}{\tau} : \Delta \underset{\sim}{e} + \frac{1}{2} \Delta \underset{\sim}{t}^T : \Delta \underset{\sim}{e}$$

$$= (\underset{\sim}{\tau} + \frac{1}{2} \Delta \underset{\sim}{t}^T) : \Delta \underset{\sim}{e} \equiv \Delta W \text{ (say)}. \tag{94}$$

Thus ΔW is the total stress-working density increment during the time interval t to $(t+\Delta t)$. As shown in [14] and [15] we may write ΔW in an alternate form, using the conjugate variables $\Delta \underset{\sim}{s}$ and $\Delta \underset{\sim}{\varepsilon}$, where $\Delta \underset{\sim}{s}$ is the incremental second Piola-Kirchhoff stress such that $(\underset{\sim}{\tau} + \Delta \underset{\sim}{s})$ is the total second Piola-Kirchhoff stress at time $(t+\Delta t)$ as referred to the configuration at t. Thus:

$$\Delta W = (\underset{\sim}{\tau} + \frac{1}{2} \Delta \underset{\sim}{t}^T) : \Delta \underset{\sim}{e}$$

$$\equiv (\underset{\sim}{\tau} + \frac{1}{2} \Delta \underset{\sim}{s}) : \Delta \underset{\sim}{\varepsilon} + \underset{\sim}{\tau} : (\Delta \underset{\sim}{e}^T \cdot \Delta \underset{\sim}{e})$$

$$\approx (\underset{\sim}{\tau} + \frac{1}{2} \Delta \underset{\sim}{s}) : \Delta \underset{\sim}{\varepsilon}. \tag{95}$$

Recall that $\underset{\sim}{\tau}$ and $\Delta\underset{\sim}{s}$ are both symmetric, and the symmetric $\Delta\underset{\sim}{\varepsilon}$ is the incremental strain between times t and t + Δt as referred to the configuration at t. Using the additive decomposition:

$$\Delta\underset{\sim}{\varepsilon} = \Delta\underset{\sim}{\varepsilon}_e + \Delta\underset{\sim}{\varepsilon}_p \tag{96}$$

where the subscripts e and p refer to "elastic" and "plastic" parts respectively, we see that:

$$\underset{\sim}{\tau} : \underset{\sim}{e} + \Delta U \equiv \Delta W = (\underset{\sim}{\tau} + \frac{1}{2} \Delta\underset{\sim}{s}) : (\Delta\underset{\sim}{\varepsilon}_e + \Delta\underset{\sim}{\varepsilon}_p)$$

$$\equiv \Delta W_e + \Delta W_p. \tag{97}$$

Where ΔW_e is "elastic" part of incremental stress-working density (per unit volume at time t) and ΔW_p is the plastically dissipated portion of incremental stress-working density. Let ΔK be the incremental kinetic energy, between t and t + dt, per unit volume in b. It is seen that:

$$\Delta K = \rho\, {}_t\underline{a}_t \cdot \Delta\underline{u}. \tag{98}$$

To simplify matters, let us consider: (i) a "mode I" 2-dimensional problem, wherein the crack-faces are free from any applied tractions or other constraints, (ii) the body forces to be zero, and (iii) the crack is along the x_1 axis. Consider a volume V*, in this 2-dimensional problem, which is enclosed by the contour Γ_{234} as in Fig. 2 . Let $\Delta\Psi_1$, ΔW_1^*, and ΔK_1^* be the incremental potential of external loads, incremental internal energy, and incremental kinetic energy, respectively, of the volume V*. It is seen that:

$$\Delta W_1^* = \int_{V*} \Delta W dv = \int_{V*} [\underset{\sim}{\tau} : \Delta\underset{\sim}{e} + \Delta U]\, dv \tag{99}$$

$$\Delta K_1^* = \int_{V*} \Delta K dv = \int_V \rho\, {}_t\underline{a}_t \cdot \Delta\underline{u} dv \tag{100}$$

$$\text{and} \quad \Delta\psi_1 = - \int_{\Gamma_{234}} \left\{ \int_0^{\Delta u_i} n_j [\tau_{ji} + \Delta t_{ji} (\mu_i)] \, d\mu_i \right\} ds \tag{101}$$

$$\equiv - \int_{\Gamma_{234}+\Gamma_{12}+\Gamma_{54}} \left\{ \int_0^{\Delta u_i} n_j [\tau_{ji} + \Delta t_{ji} (\mu_i)] \, d\mu_i \right\} ds$$

since the crack faces are traction free. To simplify the discussion further, we shall consider only the case of a stationary crack, of length "c_1". The combined incremental potential energy between t and t+dt for this cracked body is:

$$\Delta E_1 = -\Delta\psi_1 - \Delta W_1^* - \Delta K_1^* \tag{102}$$

Now, let us consider a <u>second</u> cracked-body identical in geometry to the one above, but with a crack of length "c_1+dc_1". Let the second cracked body be subjected to a load history that is identical to the one to which the first body has been subjected to. In the first body, let the cartesian coordinates measured from the crack-tip be $\zeta_1 = x_1-c_1$, and $\zeta_2=x_2$. We will consider in the second-body, a contour Γ^2_{234} which, with reference to the boundaries of the body, is identical to the contour Γ_{234} in the first body. In other words one may say, equivalently, that we are considering identical domains V* in both the bodies except that the lengths of cracks enclosed by the contours Γ^2_{234} and Γ_{234}, respectively, differ by an amount dc_1. Let ζ_k' be the cartesian coordinates centered at the crack-tip in the second body: $\zeta_1' = \zeta_1-dc_1$; $\zeta_2' = \zeta_2 = x_2$. Since both the bodies, differing in crack-lengths by dc_1, have been subject to identical load histories until time t, one may assume:

$$\text{in body 1:} \quad \left\{ \begin{array}{ll} \tau_{ij} = \tau_{ij}(\zeta_k) & ; \ \underline{a}_t = \underline{a}_t(\zeta_k) \\ \Delta\underline{u} = \Delta\underline{u}(\zeta_k) & ; \ \Delta W = \Delta W(\zeta_k) \end{array} \right\}$$

$$\text{in body 2:} \quad \left\{ \begin{array}{ll} \tau_{ij} = \tau_{ij}(\zeta_k'); & \underline{a}_t = \underline{a}_t(\zeta_k') \\ \Delta\underline{u} = \Delta\underline{u}(\zeta_k') \ ; & \Delta W = \Delta W(\zeta_k') \end{array} \right\} \tag{103}$$

The combined incremental potential energy in body 2 would be:

$$\Delta E_2 = -\Delta\psi_2 - \Delta W_2^* - \Delta K_2^* \tag{104}$$

where,

$$\Delta W_2^* = \int_{V^*} \Delta W(\zeta_k') \, dv \tag{105}$$

$$\Delta T_2^* = \int_{V^*} \rho_t \underline{a}_t \, (\zeta_k') \cdot \Delta\underline{u} \, (\zeta_k') \tag{106}$$

$$\Delta\psi_2 = -\int_{\Gamma_{234}+\Gamma_{12}+\Gamma_{54}} \left\{ \int_0^{\Delta u_i(\zeta_k')} n_j \, [\tau_{ji}(\zeta_k') + \Delta t_{ji}(\mu_i)] \, d\mu_i \right\} ds \tag{107}$$

Integrals in Eqs. (105-107) are identical to those in Eqs. (99-101), respectively, except that the limits of integration which are w.r.t. ζ_k' in Eqs. (105-107) are different from those w.r.t. ζ_k in Eqs. (99-101). By using arguments analogous to those employed earlier in connection with finite elasticity, it can be shown that:

$$\Delta W_2^* - \Delta W_1^* = -\int_{\Gamma_{234}} n_1 dc_1 \Delta W ds. \tag{108}$$

Note that singularities, if any, in ΔW near the crack-tip have been permitted in both bodies. Since the crack is assumed to be stationary, one may assume that neither $\underline{a}_t$ nor $\Delta\underline{u}$ are singular, and write:

$$\Delta T_2^* - \Delta T_1^* = -\int_{V^*} \frac{\partial}{\partial\zeta_1} (\rho \underline{a}_t \cdot \Delta\underline{u}) \, dc_1 dv \tag{109}$$

$$\Delta\underline{u}(\zeta_k') = \Delta\underline{u} \, [(\zeta_1 - dc_1), \zeta_2] = \Delta\underline{u}(\zeta_k) - \frac{\partial\Delta\underline{u}}{\partial\zeta_1} dc_1 \tag{110}$$

Finally,

$$\Delta\psi_2 - \Delta\psi_1 = +\int_{\Gamma_{234}+\Gamma_{12}+\Gamma_{45}} n_j \, [\frac{\partial\tau_{ji}}{\partial\zeta_1} \Delta u_i + (\tau_{ji} + \Delta t_{ji}) \frac{\partial\Delta u_i}{\partial\zeta_1}] dc_1 ds. \tag{111}$$

Thus, the difference in incremental total energy, in the time interval t to t+dt, between the two bodies is:

$$\frac{\Delta E_2 - \Delta E_1}{dc_1} = - \int_{\Gamma_{234}+\Gamma_{12}+\Gamma_{45}} n_j \left[\frac{\partial \tau_{ji}}{\partial \zeta_1} \Delta u_i + (\tau_{ji}+\Delta t_{ji}) \frac{\partial \Delta u_i}{\partial \zeta_1}\right] ds$$

$$+ \int_{\Gamma_{234}} N_1 \Delta W dv + \int_{V*} \frac{\partial}{\partial \zeta_1} (\rho \underline{a}_t \cdot \Delta \underline{u}) dv. \qquad (112)$$

However,

$$\int_{\Gamma_{234}+\Gamma_{12}+\Gamma_{45}} n_j \frac{\partial \tau_{ji}}{\partial \zeta_1} \Delta u_i = \int_{V*} (\tau_{ji,1} \Delta u_i)_{,j} dv$$

$$= \int_{V*} [(\tau_{ji,j})_{,1} \Delta u_i + \tau_{ji,1} \Delta u_{i,j}] dv \qquad (113)$$

In writing Eq. (113), it is assumed that τ_{ji} is non-singular near the blunted crack-tip in an elastic-plastic material, for which stress saturates to a finite value even for large strains. It is recognized, that at time t the linear momentum balance condition is:

$$\tau_{ji,j} = \rho (a_i)_t \qquad (114)$$

or $$(\tau_{ji,j})_{,1} = [\rho (a_i)_t]_{,1}. \qquad (114a)$$

Using Eqs. (114a) and (113) in Eq. (112), it is seen that:

$$\frac{\Delta E_2 - \Delta E_1}{dc_1} = \int_{\Gamma_{234}} \left[n_1 \Delta W - n_j (\tau_{ji}+\Delta t_{ji}) \frac{\partial \Delta u_i}{\partial \zeta_1}\right] ds$$

$$+ \int_{V*} \left[-\tau_{ji,1} \Delta u_{i,j} + \rho (a_i)_t \frac{\partial \Delta u_i}{\partial \zeta_1}\right] dv. \qquad (115)$$

Comparing Eq. (115) with the extreme left hand side of Eq. (93), it is seen that

$$\frac{\Delta E_2 - \Delta E_1}{dc_1} = \Delta T_1$$

where ΔT_1 is the x_1 component of the path-integral vector defined in Eq. (93). Thus, in general, it appears that the ΔT integral of Eq. (93) has the meaning: it is the difference in combined incremental potential energy (per unit crack length difference) in the time interval t to t+dt of two bodies which are identical in shape and load

history, but differ in crack lengths by 'dc_k'.

Note that in the derivation of Eq. (115), no assumption has been made concerning any loading/unloading conditions within V*. We now consider certain cases of cracks in rate-sensitive inelastic materials.

Rate-Sensitive Inelasticity:

A rate-sensitive constitutive law of considerable generality, as given by Perzyna [11],can be written for finite-strains, when an associative flow-rule is used, as:

$$\dot{\underset{\sim}{\varepsilon}}^a = \gamma\langle\phi(F)\rangle\frac{\partial F}{\partial \underset{\sim}{\sigma}} \tag{116}$$

where <> denotes a specific function, such that $\langle\phi\rangle = \phi(F)$ for $F>0$, and $\phi=0$ for $F\leq 0$. The parameter γ is called the fluidity parameter and $\dot{\varepsilon}^a$ is the inelastic strain rate. Various forms of ϕ were reviewed in [11]. For the Hencky-Mises-Huber type yield criterion, one can define F to be:

$$F = [3J_2(\underset{\sim}{\sigma}')]^{\frac{1}{2}} - F_o \equiv \sigma_{eq} - F_o = (\tfrac{3}{2}\underset{\sim}{\sigma}':\underset{\sim}{\sigma}')^{\frac{1}{2}} - F_o \tag{117}$$

where σ_{eq} is the equivalent Kirchhoff stress. A simple choice for $\phi(F)$ can be:

$$\phi(F) = F^n. \tag{118}$$

If in Eq. (117), one sets $F_0\equiv 0$, the viscoplastic strain-rate relation (116) leads to:

$$\dot{\varepsilon}^a_{ij} = \frac{3}{2}\gamma(\sigma_{eq})^{n-1}\sigma'_{ij}. \tag{119}$$

Defining the "equivalent" creep strain rate as:

$$\dot{\varepsilon}_{eq} = (\frac{2}{3}\,\dot{\varepsilon}^a_{ij}\dot{\varepsilon}^a_{ij})^{\frac{1}{2}} \tag{120}$$

it is seen from Eq. (119) that

$$\dot{\varepsilon}_{eq} = \gamma(\sigma_{eq})^n$$

which is the well-known power-law for steady-state creep. If $\hat{\dot{\sigma}}$ is the co-rotational stress-rate ($\equiv\Delta\hat{\sigma}/\Delta t$) the constitutive law can, in general, be writen as:

$$\hat{\dot{\underset{\sim}{\sigma}}} = \underset{\approx}{L}_e : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^a) \tag{122}$$

where $\underset{\approx}{L}_e$ is the tensor of instantaneous elastic modulii, and $\dot{\underset{\sim}{\varepsilon}}^a$ is the strain-rate under viscoplasticity, or creep, respectively, as given in Eq. (116) or (119). Thus,

$$\Delta\hat{\sigma} = \underset{\approx}{L}_e : \Delta\underset{\sim}{\varepsilon} - \underset{\approx}{L}_e : \int_{t_I}^{t_I+\Delta t} \dot{\underset{\sim}{\varepsilon}}^a dt. \tag{123}$$

From Eq. (123), it is seen that a potential ΔV for an elasto-viscoplastic solid, analogous to that in Eq. (73), can be constructed [16]. Using Eq. (77), a potential ΔU for $\Delta\underset{\sim}{t}$, in the case of *viscoplasticity or creep, analogous* to that in Eq. (79), can be constructed. The main points to be noted however, are: (i) in the case of an elasto-viscoplastic solid ΔU will be an explicit function of position of the material particle, since $<\phi(F)>$ is non-zero or zero (depending on $F>0$ or ≤ 0) at a given location; (ii) in the case of creep ΔU will not be an explicit function of location of the material particle.

From the above discussion, it is evident that in the case of cracks in elasto-viscoplastic solids, an incremental integral vector, $\Delta\underline{T}$, entirely equivalent to Eq. (93) can be defined. Moreover, it is also evident that the $\Delta\underline{T}$ integral in the case of viscoplasticity is path-independent in the same limited sense as discussed earlier in connection with rate-independent elasto-plasticity.

On the other hand, in the case of creep, described by a constitutive law of type (119), since ΔU is not an explicit function of position of the

material particle, we have the result, analogous to the case of finite elasticity, that $\underset{\sim}{R}=0$ in Eq. (93).

Thus, in the case of creep, we have, for instance in a 2-dimensional case, the following result instead of Eq. (93):

$$\int_{\Gamma_{234}} [\underset{\sim}{n}(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U) - \underset{\sim}{n} \cdot \langle(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot \Delta\underset{\sim}{e}\rangle]ds + \underset{\varepsilon\to 0}{Lt} \left\{\int_{V-V_\varepsilon} [-\underset{\sim t}{\nabla}\underset{\sim}{\tau}:\Delta\underset{\sim}{e}\right.$$

$$- \rho_t\,(\underset{\sim}{f}-\underset{\sim}{a}) \cdot \Delta\underset{\sim}{e}]dv + \int_{\Gamma_{12}} \underset{\sim}{n}^+ [(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U)^+ - (\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U)^-]ds$$

$$\left.- \int_{\Gamma'_{12}} [(\underset{\sim}{\bar{t}}.\Delta\underset{\sim}{e})^+ + (\underset{\sim}{\bar{t}}.\Delta\underset{\sim}{e})^-]ds - \int_{\Gamma''_{12}} \underset{\sim}{n}^+[\langle(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot \Delta\bar{\underset{\sim}{e}}\rangle^+ - \langle(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}).\Delta\bar{\underset{\sim}{e}}\rangle^-]\,ds\right\}$$

$$= \underset{\varepsilon\to 0}{Lt} \int_{\Gamma_\varepsilon} [\underset{\sim}{n}(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U) - \underset{\sim}{n} \cdot \langle(\underset{\sim}{\tau}+\Delta\underset{\sim}{t}) \cdot \Delta\underset{\sim}{e}\rangle]ds$$

$$\equiv \Delta\underset{\sim}{T}_c \tag{124}$$

where $\Delta\underset{\sim}{T}_c$, the incremental vector for the case of a crack in a creeping material, is once again, by definition, the non-zero limit (in the limit $\varepsilon\to 0$) of the contour integral on Γ_ε as in Eq. (124). The far field evaluation of $\Delta\underset{\sim}{T}_c$ should be performed as per the contour/volume integrals appearing on the extreme L.H.S. of Eq. (124). Once again, it is evident that since ΔU is not an explicit function of location, the integral appearing on the extreme L.H.S. of Eq. (124) is strictly path-independent (eventhough the enclosed volumes $V-V_\varepsilon$ change). Further it is seen that the $\Delta\underset{\sim}{T}_c$ integral in the case of creep, characterizes the severity of crack-tip conditions.

Consider the application of Eq. (124) to the special case: (i) body forces including inertia are negligible, (ii) the crack-faces are free from any prescribed conditions, (iii) the crack is aligned with the x_1 axis and that we are interested only in the x_1 component of the vector $\Delta\underset{\sim}{T}_c$. Under these restrictions, Eq. (124) becomes:

$$\int_{\Gamma_{234}} [n_1(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U) - n_j\ (\tau_{ji}+\Delta t_{ji})\ \frac{\partial \Delta u_i}{\partial x_1}]ds - \underset{\varepsilon\to 0}{Lt} \left\{\int_{V-V_\varepsilon} \frac{\partial \tau_{ji}}{\partial x_1}\frac{\partial \Delta u_i}{\partial x_j}\, dv\right\}$$

$$= \underset{\varepsilon\to 0}{Lt} \int_{\Gamma_\varepsilon} [n_1(\underset{\sim}{\tau}:\Delta\underset{\sim}{e}+\Delta U) - n_j\ (\tau_{ji}+\Delta t_{ji})\ \frac{\partial \Delta u_i}{\partial x_1}]ds$$

$$\equiv (\Delta T_1)_c. \tag{125}$$

Once again, it can be shown that $(\Delta T_1)_c$ is the difference in incremental total energies during t and t+Δt of two bodies identical in shape except differing in crack-lengths by 'dc_1', both of which have been subject to identical load histories upto time t.

We observe that Landes and Begley [10], and Goldman and Hutchinson [9] introduced a parameter c* defined as:

$$c^* = \int_\Gamma [W^*dy - T_i\ \frac{\partial \dot{u}_i}{\partial x}]ds \tag{126}$$

to correlate creep crack growth rates in "Mode I" where Γ is a far-field contour [analogous to Γ_{234} in Eq. (125)] and

$$W^* = \int_0^{\dot{\varepsilon}_{mn}} \sigma_{ij}\, d\dot{\varepsilon}_{ij} \tag{127}$$

we note that the above parameter was introduced in [9, 10] based primarily on the observation of the essential similarity between the pure power-law hardening type (deformation-theory-of plasticity) constitutive relations between the strain and stress on the one hand, and the steady-state creep relation of the power-law type between the strain-rate and stress on the other.

The present parameter $(\Delta T_1)_c$, of Eq. (125), however, accounts for elastic as well as creep strains (using the decomposition, $\Delta\underset{\sim}{\varepsilon} = \Delta\underset{\sim}{\varepsilon}^e+\Delta\underset{\sim}{\varepsilon}^c$ where superscripts e and c stand for "elastic" and "creep"), finite deformations, and is valid even under non-steady creep conditions. The path-independent nature of $(\Delta T_1)_c$ is in Eq. (125) is evident from previous discussion. With the <u>definition</u> of

$(\Delta T_1)_c$ as the limit $\epsilon \to 0$ of the contour integral Γ_ϵ, the local nature of $(\Delta T_1)_c$, i.e., its role as a parameter quantifying the crack-tip conditions is evident.

Under steady-state creep conditions, it is seen that $(\Delta T_1)_c$ of Eq. (125) reduces to $(\Delta T_1)_c^s$ where,

$$(\Delta T_1)_c^s \equiv \underset{\epsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\epsilon} [n_1(\underset{\sim}{\tau} : \Delta \underset{\sim}{\dot{e}}) - n_j(\tau_{ji}) \frac{\partial \Delta u_i}{\partial x_i}] \, ds. \tag{128}$$

or $$(\dot{T}_1)_c^s = \underset{\epsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\epsilon} [n_1(\underset{\sim}{\tau} : \underset{\sim}{\dot{e}}) - n_j(\tau_{ji}) \frac{\partial \dot{u}_i}{\partial x_1}] \, ds \tag{129}$$

on the other hand, C* of Eq. (126) can equivalently be written as:

$$C^* \equiv \underset{\epsilon \to 0}{\mathrm{Lt}} \int_{\Gamma_\epsilon} [W^* dy - T_i \frac{\partial \dot{u}_i}{\partial x}] \, ds \tag{130}$$

where

$$W^* = \int_0^{\dot{\epsilon}_{mn}} \sigma_{ij} d\dot{\epsilon}_{ij} \tag{130b}$$

Thus, even under steady-state conditions, the present $(\dot{T}_1)_c^s$ differs from C* of [9,10], in that; while $\underset{\sim}{\tau} : \underset{\sim}{\dot{e}}$ occurs in the former, W* occurs in the later. It is noted that while $\underset{\sim}{\tau} : \underset{\sim}{\dot{e}}$ has the physical interpretation as the stress-working rate, W* is simply a mathematical potential for σ_{ij} under steady-state creep conditions. Thus while $(\dot{T}_1)_c^s$ still has a physical interpretation as discussed earlier, C* does not, in general, have a physical meaning. These and other topics relevant to the use of $(\Delta T_1)_c$ and C* are more elaborately discussed in [17].

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References:

[1] Eshelby, J.D., "The Continuum Theory of Lattice Defects", Solid State Physics, vol. III, Academic Press, pp 79-144, (1956).

[2] Rice, J R., "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks", Journal of Applied Mechanics, vol. 35, pp 379-386 (1968).

[3] Knowles, J.K., and Sternberg, E., "On a Class of Conservation Laws in Linearized and Finite Elastostatics", Archive for Rational Mechanics and Analysis, vol. 44, No. 3, pp 187-211 (1972).

[4] Budiansky, B., and Rice, J.R., "Conservation Laws and Energy Release Rates", Journal of Applied Mechanics, vol. 40, No. 1, pp 201-203 (1973).

[5] Rice, J.R., "Elastic-Plastic Fracture Mechanics", In The Mechanics of Fracture, ASME-AMD vol. 19 (F. Erdogan, Ed.) ASME, N.Y., pp 23-54 (1976).

[6] Paris, P.C., and Zahalak, G.I., "Progress in Elastic-Plastic Fracture Mechanics and its Applications", In Nonlinear and Dynamic Fracture Mechanics, ASME-AMD vol. 35 (N. Perrone, and S.N. Atluri, Eds.), ASME, N.Y., pp 125-234 (1979).

[7] Freund, L.B., "Energy Flux into the Tip of an Extending Crack in an Elastic Solid", Journal of Elasticity, vol. 2, pp 341-349 (1972).

[8] Freund, L.B., "Dynamic Crack Propagation", In The Mechanics of Fracture, ASME-AMD, vol. 19, (F. Erdogan, Ed.) ASME, N.Y., pp 105-134 (1976).

[9] Goldman, N.L., and Hutchinson, J.W., "Fully-Plastic Crack Problems: The Center Cracked Strip under Plane Strain", Intl. Jnl. of Solids and Structures, vol. 11, No. 5, pp 575-592 (1975).

[10] Landes, J.D., and Begley, J.A., "A Fracture Mechanics Approach to Creep Crack Growth", in Mechanics of Crack Growth, ASTM STP 590, American Society for Testing and Materials, pp 128-148 (1976).

[11] Perzyna, P., "Fundamental Problems in Visco-Plasticity", Advances in Applied Mechanics, vol. 9, Academic Press, pp 243-377 (1966).

[12] Atluri, S.N., and Murakawa, H., "On Hybrid Finite Element Models in Non-linear Solid Mechanics", in Finite Elements in Nonlinear Mechanics, vol. 1 (P.G. Bergan, et al, Eds.) Tapir Press, Norway, pp 3-41 (1977).

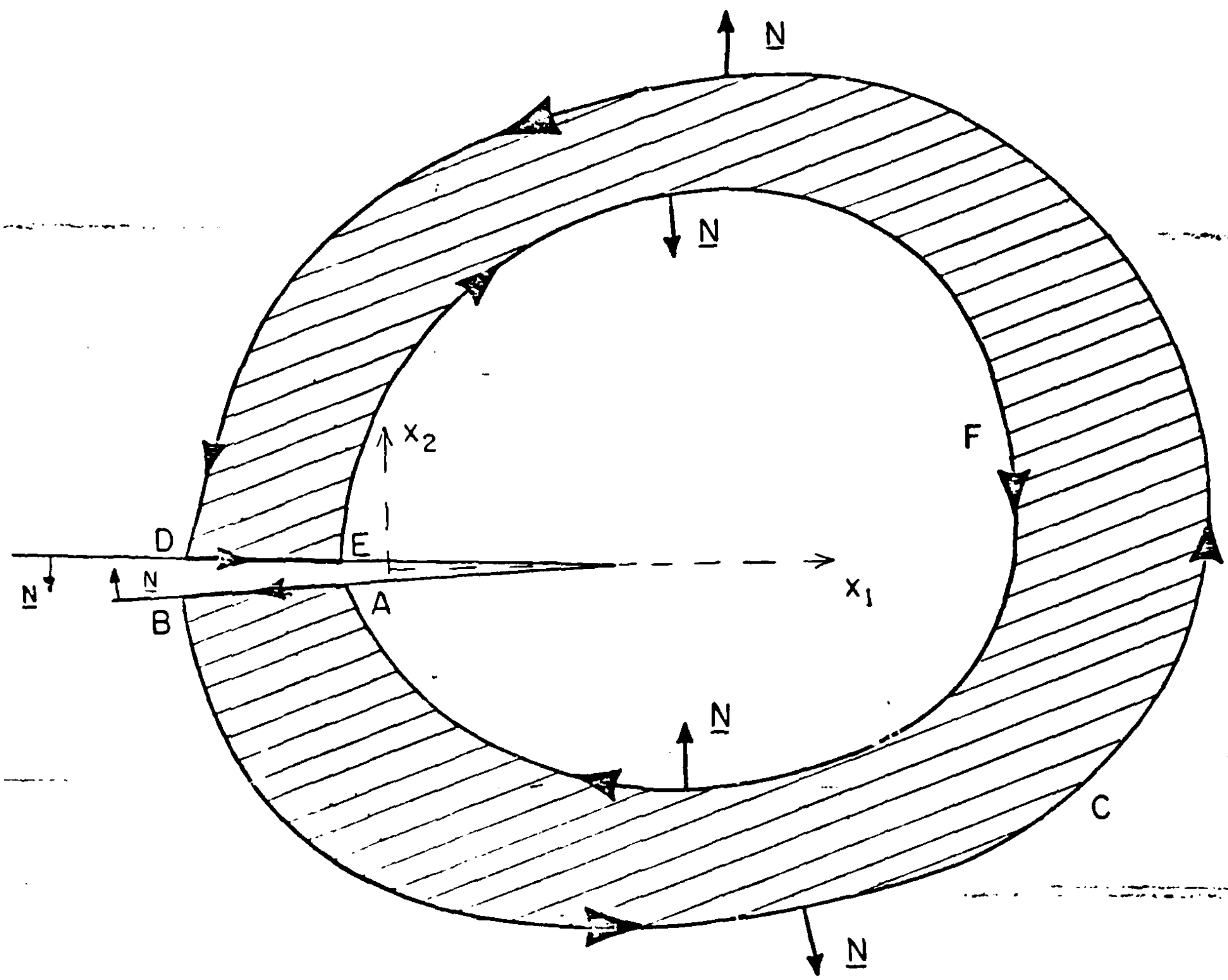
[13] Herrmann, A.G., and Herrmann, G., "On Energy-Release Rates for a Plane Crack", Division of Applied Mechanics, Stanford University (Undated manuscript) 21 pp (Received, Jan. 1981).

[14] Hill, R., "Eigenmodal Deformations in Elastic/Plastic Continua", Jnl. Mech. and Phys. Solids, vol. 15, pp 371-386 (1967).

[15] Atluri, S.N., "On Some New General and Complementary Energy Theorems for the Rate Problems of Finite Strain, Classical Elastoplasticity", Journal of Structural Mechanics, vol. 8, No. 1, pp 61-92 (1980).

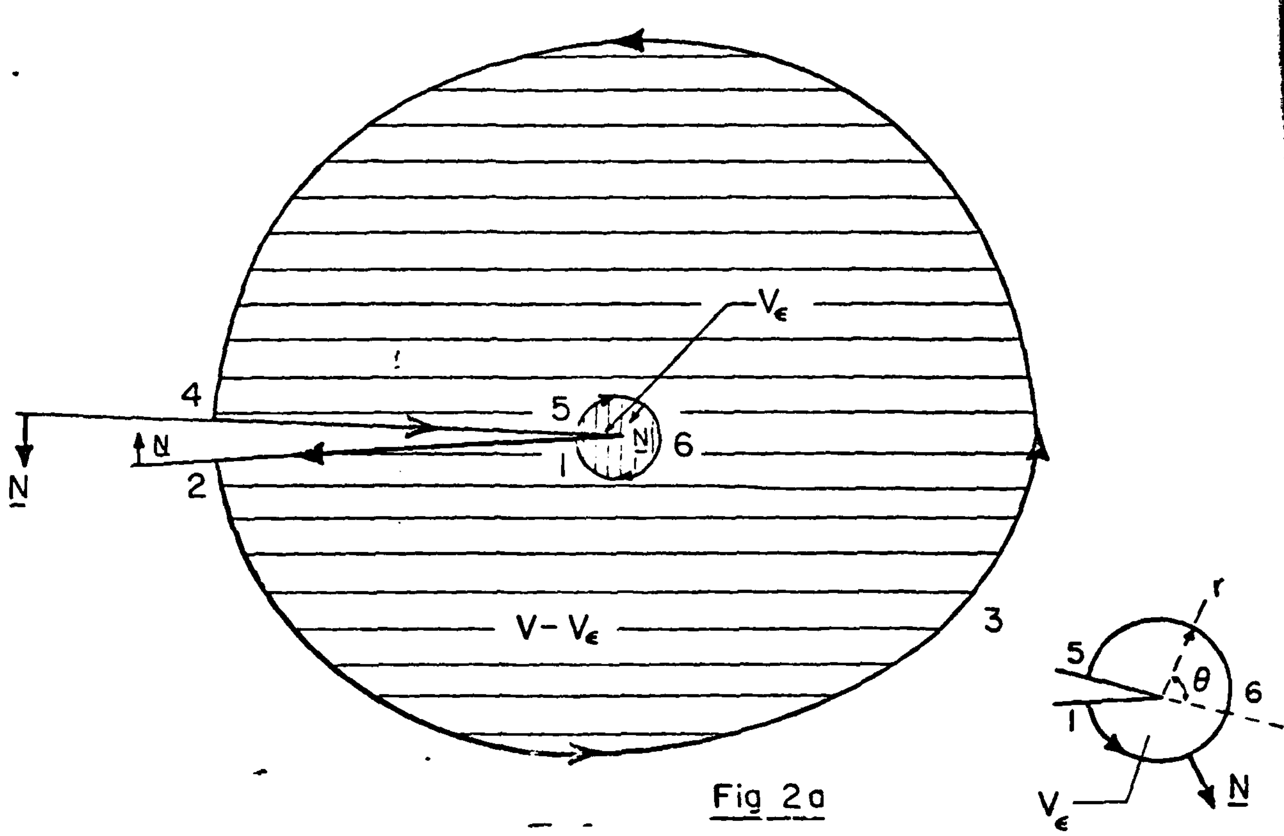
[16] Atluri, S.N., "New General and Complementary Energy Theorems, Finite Strain, Rate-Sensitive Inelasticity, and Finite Elements", to appear in Proc. U.S.-Europe Workshop on Finite Elements in Nonlinear Structural Mechanics, Ruhr-University, Bochum, W. Germany, July 1980.

[17] Stonesifer, R.B., and Atluri, S.N., "A Study on the use of $(\Delta \underline{T})_c$ and C* Integrals in Fracture Analysis Under Non-Steady-Creep conditions", Manuscript submitted to Engineering Fracture Mechanics.



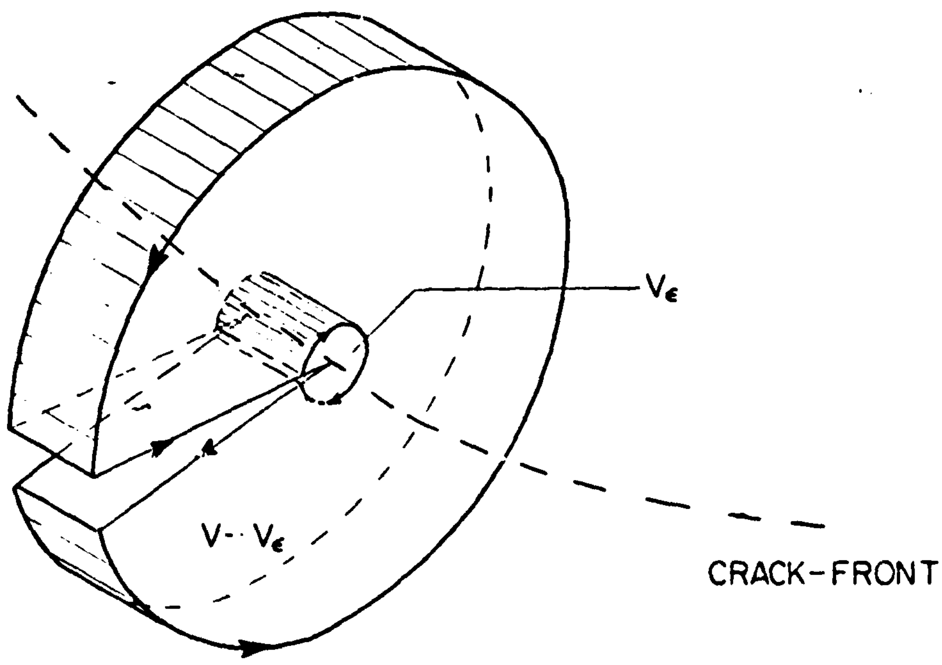
$$S_i = \int_{\Gamma_{EFA}} + \int_{\Gamma_{BCD}} \quad ; \quad S_F' + S_t = \int_{\Gamma_{AB}} + \int_{\Gamma_{DE}}$$

Fig 1



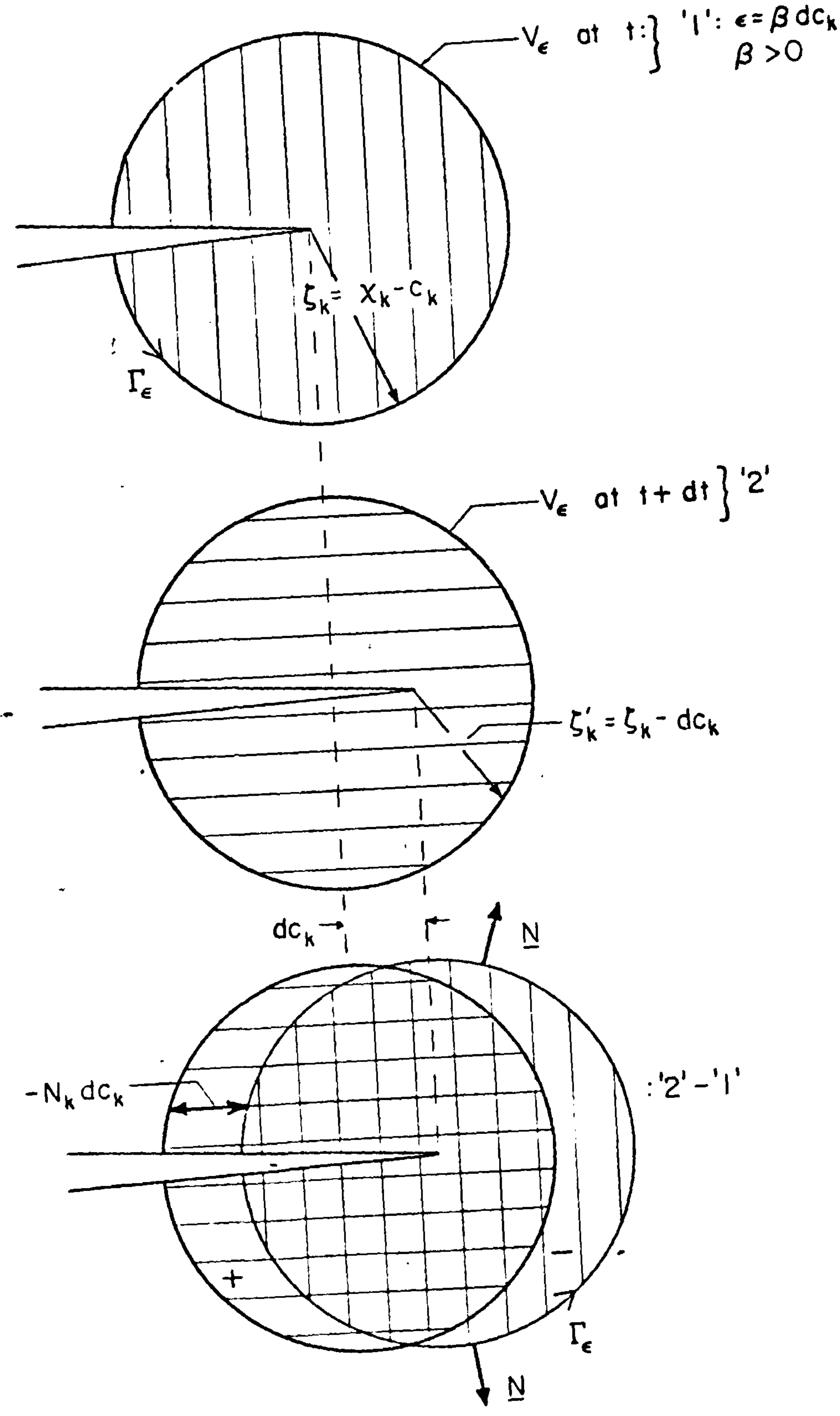
V_ϵ
4
5
N
6
N
N
2
1
$V - V_\epsilon$
3
r
5
θ
6
1
N
V_ϵ

Fig 2a



V_ϵ
$V - V_\epsilon$
CRACK-FRONT

Fig 2b



V_ϵ at t: } '1': $\epsilon = \beta\, dc_k$
$\beta > 0$
$\zeta_k = X_k - c_k$
Γ_ϵ
V_ϵ at $t + dt$ } '2'
$\zeta'_k = \zeta_k - dc_k$
dc_k
$\underline{N}$
$-N_k\, dc_k$
: '2' − '1'
+
−
Γ_ϵ
$\underline{N}$

Fig. 3

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In this paper, certain path-independent integrals, of relevance in the presence of cracks, in elastic and inelastic solids are considered. The hypothesized material constitutive properties include: (i) finite and infinitesimal elasticity, (ii) rate-independent incremental flow theory of elasto-plasticity, and (iii) rate-sensitive behaviour including elasto-viscoplasticity, and creep. In each case, finite deformations are considered, along with the effects of body forces, material acceleration, and arbitrary traction/displacement conditions on the crack-face. Also the physical interpretations of each of the integrals

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either in terms of crack-tip energy release rates or simply energy-rate differences in two comparison cracked-bodies are explored. Several differences between the results in the present work and those currently considered well established in literature are pointed out and discussed.